MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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ROYAL ASTRONOMICAL SOCIETY

Vol. 100 No. 1

MEETING OF 1939 NOVEMBER 10

Professor H. C. Plummer, President, in the Chair

At a Special General Meeting of Fellows held at 14h. 15m., before the Ordinary Meeting, the following additional Bye-laws, which were recommended by the Council as being necessary to meet the requirements of the present emergency, and to enable Council to take appropriate action in the best interests of the Society, were proposed severally by Mr. D. H. Sadler, Secretary, on behalf of the Council :—

> No. 87. Notwithstanding the present Bye-laws, the Council may, at any time during the present emergency when the normal procedure of calling a Special General Meeting is impracticable, suspend or abrogate such Bye-laws as may appear necessary or desirable in the interests of the Society.

This was seconded by Dr. Atkinson and, after discussion, was carried by a majority vote.

SECTION V

> No. 26A. Notwithstanding the preceding Bye-laws, any Fellow of the Society who is on Active Service with H.M. Forces may, if he so desires, be excused from payment of his Annual Contribution until the end of the "emergency period" (*i.e.* the end of the year in which peace is declared). He shall not during that period forfeit his Fellowship, and his name shall not be suspended as a defaulter by reason of non-payment of such Annual Contribution, as provided in Bye-law 26 ; but he shall not receive the publications of the Society until he resumes the payment of his Annual Contribution.
>
> At the end of the emergency period he shall have the option of resuming his Annual Subscription from that date, or any previous

date, receiving the publications of the Society from the date of resumption; or of resigning his Fellowship without any liability for arrears of Subscription beyond those outstanding at the date of his application to be excused from payment of his Annual Contribution.

This was seconded by the Treasurer, and carried without opposition.

SECTION VIII

No. 41A. For the duration of the present lighting restrictions, and during the present emergency, business at ordinary meetings of the Society shall commence at such earlier hour as Council may direct.

This was seconded by Dr. Atkinson and carried without opposition.

The Ordinary Meeting of the Society then followed at 14h. 45m.

The President read the following Resolutions of Condolence which had been passed by the Council that afternoon, with reference to the deaths of Dr. A. C. D. Crommelin and Mr. H. P. Hollis, and the Meeting stood in silence as a tribute to their memory :—

"The Council wishes to record the great loss that the Society in particular and Astronomy in general has suffered by the death of Andrew C. de la C. Crommelin. As President, as Secretary, as member of Council and as a Fellow for fifty-one years he had served the Society truly and well. Not only will his specialised knowledge of Comets and Minor Planets be sadly missed, but the friendly criticism and advice which he gave to all who asked will not easily be replaced. His death leaves a gap that will be difficult to fill."

"The Council wishes to record its deep regret on the death of Henry Park Hollis. As a Fellow of the Society for fifty-five years, as a member of Council and as a fervent lover of Astronomy, he made himself a special place in the regard of all who knew him; his wide knowledge of astronomy and astronomers gathered over a long life devoted to the subject, his keen interest and remarkable memory, and above all his courage in the face of physical disability, will be sadly missed."

The election by the Council of the following Fellows was duly confirmed :—

Miss Lydia Alberta Brown, 303 Bath Road, Hounslow West, Middlesex;
H. W. S. Massey, Professor of Mathematics, University College, London, W.C. 2;
Allan Richard Dalhousie Ramsay, M.A., Marlborough College, Wilts; and
Rev. Wilfred Rees Wright, M.Sc., 21 St. Mark's Road, Salisbury, Wilts.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Dr. A. E. Benfield, Thompson Physical Laboratory, Williams College, Williamstown, Mass., U.S.A. (proposed by R. Stoneley);

Douglas Charles Berry, Lands and Survey Department, Invercargill, New Zealand (proposed by M. Geddes);

William Edward Browne, F.S.I., F.R.G.S., c/o Anglo-Iranian Oil Co. Ltd., Britannic House, Finsbury Circus, London, E.C. 4 (proposed by L. J. Comrie);

Hugh Ernest Butler, B.A., Emmanuel College, Cambridge (proposed by F. J. M. Stratton);

Edward Hope-Jones, B.A., c/o H.M. Office of Works, Contract No. 4, R.O.F., Glascoed, Usk, Mon. (proposed by L. E. Lefèvre);

David G. Kendall, University Observatory, Oxford (proposed by H. H. Plaskett);

Thomas James Latus, 7 Willoughby Road, Hampstead, N.W. 3 (proposed by C. C. L. Gregory);

Edward Hubert Linfoot, D.Phil., 13 Alexander Road, Clifton, Bristol 8 (proposed by C. R. Burch);

John Felix de Lisle, M.Sc., 16 View Road, Mount Eden, Auckland, S. 1, New Zealand (proposed by K. E. Bullen);

Rev. Charles Linaker Lomas, The Rectory, Badlesmere, near Faversham, Kent (proposed by U. L. Johnstone);

Ernest Henry Noon, 23 Belltrees Grove, Streatham, London, S.W. 16 (proposed by A. G. Fellows);

Roger W. Pring, B.Sc., 34 Village Road, Finchley (proposed by C. C. L. Gregory);

J. J. Shaw, C.B.E., M.Sc., Sunnyside, West Bromwich (proposed by H. Jeffreys);

Arthur Albert Smith, B.Sc., Sevenoaks School, Kent (proposed by A. P. Rollett);

Dr. Lyman Spitzer, Jr., Sloane Physics Laboratory, Yale University, New Haven, Conn., U.S.A. (proposed by H. Shapley);

Leslie Granville Taylor, 17 Cawdor Crescent, Boston Manor, London, W. 7 (proposed by W. Denne); and

Eric Bell Walton, A.M.I.C.E., A.R.I.B.A., 42 Hainault Avenue, Westcliff, Essex (proposed by P. Murden).

One hundred and seventy-four presents were announced as having been received since the last meeting, including :—

Hamburg Observatory, Spektral-Durchmusterung von Milchstrassenfeldern. Teil I.

Hamburg Observatory, Viertes Bergedorfer Sternverzeichnis 1950·0;

International Astronomical Union, Transactions, vol. 6;

H. Spencer Jones, The Earth as a Clock (Halley Lecture, 1939);

H.M. Nautical Almanac Office, Planetary Co-ordinates for the Years 1940–1960, referred to the equinox of 1950·0;

H.M. Nautical Almanac Office, Seven-figure Tables for every Second of Time.

STARK EFFECT AND DAMPING FACTOR IN THE FRAUNHOFER SPECTRUM

C. W. Allen, D.Sc.

Studies of the curve of growth for Fraunhofer lines have led consistently to the result that the damping factor is considerably greater than the classical radiation damping value of $8\pi^2e^2\nu^2/3mc^3$. In cases where the quantum theory provides a value for the radiation damping this is also smaller than the observed value. Unsöld * has gone some way towards explaining the discrepancy by showing that in many cases the quadratic Stark effect is sufficient to provide very considerable collision damping. The theory gives reasonable results for the damping factor of the *Mg* ^{1}PD series, and the *Na* and *Ca* resonance lines.† The Stark effect measurements of iron lines by Panter and Foster ‡ have made it possible to test the theory more directly by comparing the Stark effect for a number of lines with the damping constant determined from the curve of growth. The comparison leads to an independent determination of the electron pressure in the Sun's reversing layer.

Observations.—For the curve of growth it is necessary to obtain (*a*) the equivalent breadth of a number of Fraunhofer lines, and (*b*) the corresponding relative values of N_lf (number of atoms in the lower level × oscillator strength).

The solar equivalent breadths published already § are not sufficiently accurate nor homogeneous for the purpose, and these have all been re-measured. The lines used and the values now obtained for the equivalent breadths (W) are given in Table I.

The values of N_lf were determined by photographic photometry of lines from an iron arc of 5 amps, 200 volts, and 3 mm. pole separation. It will appear that the excitation temperature of the arc under these conditions is the same as that of the Sun. The arc spectra were photographed through a Utrecht step-wedge, and it was found possible to measure intensities with a range of 1 : 500 with sufficient accuracy. The faintest lines were only just distinguishable from the background. The majority of lines used for the curve of growth have a low-excitation-potential in the neighbourhood of 4 volts, and it is unlikely that there would be enough absorbing atoms in such high levels to produce appreciable self-reversal. There is no visible sign of self-reversal even for the lines from much lower

* A. Unsöld, *Zs. f. Astrophys.*, **12**, 56, 1936.

† A. Unsöld, *Physik der Sternatmosphären*, 281, 1938.

‡ S. F. Panter and J. S. Foster, *Proc. Roy. Soc.*, **162**, 336, 1937.

§ C. W. Allen, *Mem. Comm. Sol. Obs.*, Canberra, **1**, No. 5, 1934; M. Minnaert and G. F. W. Mulders, *Zs. f. Astrophys.*, **1**, 192, 1930; M. G. Adam, *M.N.*, **98**, 112, 1937.

λ (Sun)	Low E.P. Volts	W Milli-A.	log $10^6 W/\lambda$	log $N_i f$	Stark Effect 100 kv./cm. cm.$^{-1}$	$10^{10} \times C^{2/3}$	a
4983·260	4·136	109	1·34	1·99	0·2	6	0·07
4983·861	4·086	120	1·38	2·12	0·2	6	0·07
4985·261	3·912	98	1·29	1·98	0·0	0	0·03
4986·230	4·199	45	0·95	0·83			
4986·915	4·238	15	0·48	0·21			
4988·957	4·136	87	1·24	1·65	0·4	9	
4990·455		28	0·75	0·58			
4991·277		86	1·24	1·66	0·4	9	
4999·122		30	0·78	0·59			
5001·872	3·865	134	1·43	2·41	0·0	0	0·05
5002·799	3·382	78	1·19	1·43			
5004·051		45	0·95	0·81			
5005·721	3·867	124	1·39	2·34	0·0	0	0·04
5006·122	2·820	200	1·60	2·72	0·0	0	(0·08)
5013·925		16	0·51	0·09			
5014·951	3·926	116	1·36	2·21	0·0	0	0·04
5016·888	4·202	42	0·92	0·75			
5019·737	3·967	17	0·53	0·24			
5023·194		33	0·82	0·58			
5023·499		27	0·73	0·50			
5027·131	4·136	100	1·30	1·94	0·5	10	0·04
5068·773	2·927	129	1·40	2·22	0·0	0	0·07
5074·755	4·202	121	1·38	2·13	1·3	19	0·07
5078·983	4·283	98	1·29	1·85	1·9	24	0·06
5079·232	2·188	106	1·32	2·08	0·0	0	0·03
5088·159		36	0·85	0·62			
5090·782	4·238	90	1·25	1·65	0·1	4	
5099·082	3·967	56	1·04	0·99			
5104·198	4·160	24	0·67	0·24			
5104·441		31	0·79	0·55			
5125·132	4·202	110	1·33	2·03	1·3	19	0·05
5127·690	0·051	17	0·52	0·14			
5129·636	3·926	42	0·91	0·71			
5131·478	2·213	84	1·21	1·57			
5133·701	4·160	157	1·49	2·45	3·0	32	0·08
5136·103		19	0·57	0·29			
5137·395	4·106	108	1·32	1·95	0·2	6	0·06
5141·748	2·414	89	1·24	1·59			
5142·532	4·238	114	1·34	1·79	0·0	0	(0·12)
5145·104	2·188	50	0·99	0·87			
5150·196		15	0·46	0·02			
5162·283	4·160	153	1·47	2·30	7·4	59	0·10
5164·554		41	0·90	0·61			
5165·417		89	1·24	1·61	0·2	6	

levels, and the agreement between arc and solar intensities * would suggest that one is justified in neglecting self-reversal for the majority of lines under consideration. In order to allow for the difference in the breadth of the lines it was necessary to multiply the intensity of the maximum by the width of the lines at half the maximum intensity. In a rich spectrum, such as the *Fe* arc where comparatively few lines are completely isolated, this is probably the most accurate method of including line breadth in intensity measurements. A colour sensitivity correction was obtained by comparing the densities at different wave-lengths of the solar spectra obtained for the purpose of checking the equivalent breadths. The exposure times for the solar spectra were much shorter than for the arc, but it is unlikely that this would appreciably affect the colour correction in a region of less than 200 A. The intensities were multiplied by λ^3 in order to give values proportional to $N_u f_{em}$, where N_u is the number of atoms in the upper level, and f_{em} is the emission oscillator strength. Then

$$N_l f = N_u \cdot \frac{g_l}{g_u} \cdot e^{(E_u - E_l)/kT} \cdot \frac{g_u}{g_l} \cdot f_{em}$$
$$= N_u \cdot f_{em} \cdot e^{(E_u - E_l)/kT},$$

where g_l and g_u are the statistical weights, E_l and E_u the excitation potentials, k the Boltzmann constant, and T the temperature. The variation of the exponential factor is small over the wave-length range 4983–5165 A., and it may be computed with sufficient accuracy by putting $T = 5000°$ K. The values of $\log N_l f$ on an arbitrary scale are given in Table I.

Table I contains also the mean Stark effect displacements for a field of 100 kv./cm. These values are taken from the work of Panter and Foster † who remark that the displacement is approximately proportional to the square of the applied field.

Curve of Growth.—The curve of growth is obtained by plotting $N_l f$ against W/λ, where W is the equivalent breadth of the solar line in wave-length units. The resulting points on the curve are shown in fig. 1. As seen from Table I, the majority of lines used have their low-excitation potential in the neighbourhood of 4 volts. A few lines of the lower E. P., however, have been purposely included, and the fact that the points for these lines lie well on the curve is an indication that the excitation temperature of the iron arc was the same as that of the Sun. It is clear that there is no need to make a correction for any difference in solar and arc temperature.

The corresponding theoretical curves of growth are shown also in fig. 1. These are taken from Unsöld's formula ("Sternatmosphären" pp. 266–268) using the value

$$R_c \cdot 2\Delta\lambda_D/\lambda = 1{\cdot}05 \times 10^{-5},$$

where R_c is the depth of strong Fraunhofer lines, and $2\Delta\lambda_D$ is the whole Doppler breadth measured from the point where the intensity is decreased

* C. W. Allen, *M.N.*, 96, 145, 1935.
† *Loc. cit.*

to $1/e$ of the maximum. The nomenclature in the following formulæ are from Unsöld's "Sternatmosphären." The value adopted is the mean between the best fit for the points on the left of fig. 1 ($1{\cdot}15 \times 10^{-5}$), and a value recently determined for *Fe* from independent considerations of the curve of growth ($0{\cdot}95 \times 10^{-5}$).

Damping Factor, Stark Effect, and Electron Pressure.—It is seen at once from fig. 1 that the Stark effect has not produced a very great scatter from the smooth curve, and it remains to be seen whether the scatter is due to

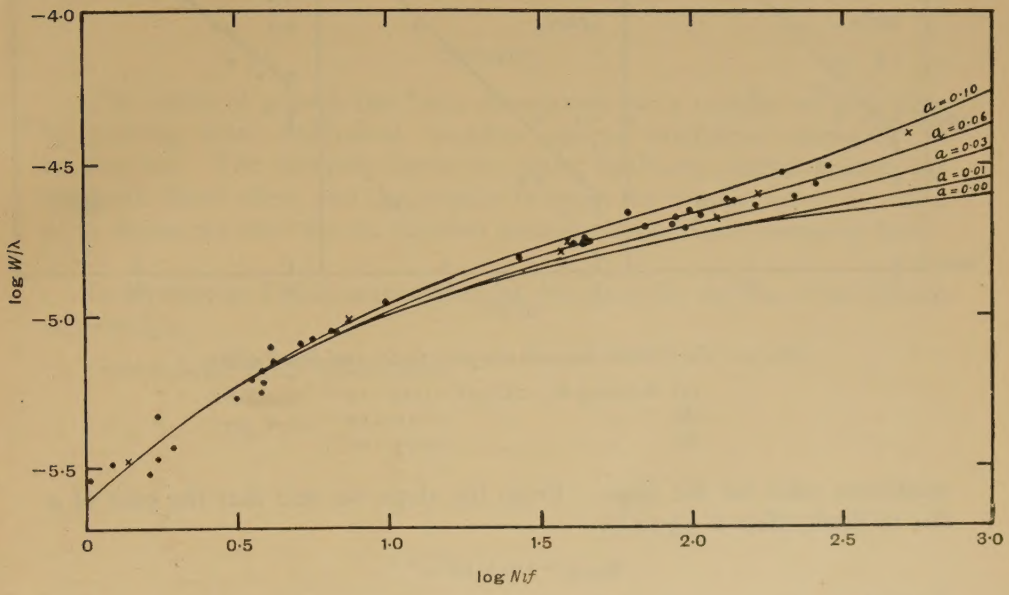

FIG. 1.—*Curve of growth.*

Points: Lines with low-E.P. of approximately 4 volts.
Crosses: Lines with low-E.P. less than 3 volts.
Continuous curve: Curve of growth from Unsöld's formula.

Stark effect or experimental error. The theoretical curve of growth is divided on the right into a number of curves corresponding to varying damping factors defined by the constant $a = \gamma/\Delta\omega_D$, where γ is the damping constant and $\Delta\omega_D$ represents the Doppler broadening $\Delta\lambda_D$ in circular frequency units. Thus each point on the right of the diagram defines a value of a, and the values so found for lines with $N_l f > 1{\cdot}7$ are given in Table I.

On the basis of Unsöld's theory ("Sternatmosphären," p. 280) the values of a should show a linear relation with $C^{2/3}$, where

$$C = 61{\cdot}5 \times 10^{-15} \times \text{displacement in cm.}^{-1} \text{ for 100 kv./cm. field.}$$

The values of $C^{2/3}$ are given in Table I, and are plotted against a in fig. 2 (a). No linear relation appears until it is found legitimate to discard the lines 5006 A. (since this strong low-level line is likely to be reversed in the arc) and 5142 A. (because of the proximity of another strong line in the Sun's

spectrum). Making these discards it seems reasonable to assume a linear relation, and the straight line drawn in fig. 2 (*a*) should represent perhaps a

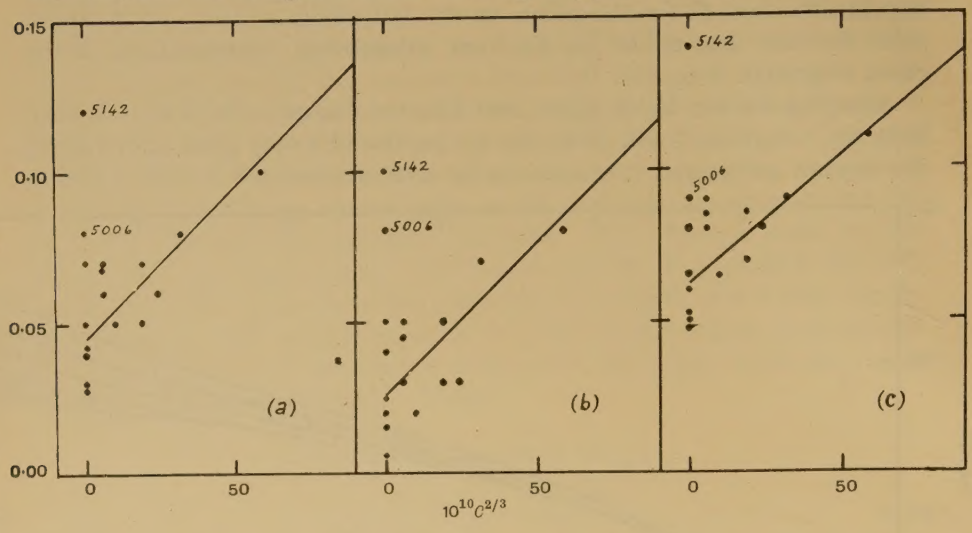

FIG. 2.—*The relation between damping factor and Stark effect.*

(*a*) *Adopting* $R_c \cdot 2\Delta\lambda_D/\lambda = 1{\cdot}05 \times 10^{-5}$.
(*b*) ,, ,, $= 1{\cdot}15 \times 10^{-5}$.
(*c*) ,, ,, $= 0{\cdot}95 \times 10^{-5}$.

maximum value for the slope. From the slope we find that the part of *a* due to Stark effect is given by

$$a_{\text{Stark}} = 1{\cdot}0 \times 10^{7} C^{2/3}.$$

To find $\gamma_{\text{Stark}} (= a_{\text{Stark}} \cdot \Delta\omega_D)$ we put as before

$$R_c \cdot 2\Delta\lambda_D/\lambda = R_c \cdot 2\Delta\omega_D/\omega = 1{\cdot}05 \times 10^{-5},$$

and adopt

$$R_c = 0{\cdot}9$$

and

$$\omega = 2\pi c/\lambda = 3{\cdot}7 \times 10^{15} \text{ sec.}^{-1},$$

giving

$$\gamma_{\text{Stark}} = 2{\cdot}2 \times 10^{17} \times C^{2/3} \text{ sec.}^{-1}.$$

Equating this to Unsöld's formula ("Sternatmosphären," 71, 14) gives

$$2^{3/2}\pi^{13/6}(RT)^{1/6} \cdot \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)^{1/6} \cdot N = 2{\cdot}2 \times 10^{17}.$$

From Unsöld's discussion approximately 0·86 of the effect is due to electrons; hence putting $R = 8{\cdot}32 \times 10^{7}$, $T = 5600°$ K., $\mu_1 = 56$, and $\mu_2 = 1/1838$ we obtain for N_e the number of electrons per cm.^{3} in the reversing layer,

$$N_e = 1{\cdot}8 \times 10^{13} \text{ per cm.}^3$$

and

$$P_e = kTN_e = 14 \text{ dynes per cm.}^2.$$

The value of 14 dynes/cm.^{2} for the electron pressure of the reversing layer is in good agreement with the lower values determined from other information ("Sternatmosphären," p. 348).

The values of a obtained from the curve of growth are very sensitive to the chosen $R_c \,.\, 2\Delta\lambda_D/\lambda$ and hence the procedure has been repeated, using, firstly, $1{\cdot}15 \times 10^{-5}$, and secondly, $0{\cdot}95 \times 10^{-5}$. The plots of a so obtained against $C^{2/3}$ are shown in fig. 2, (*b*) and (*c*). It will be seen that the slope which determines a_{Stark} is not greatly affected by varying this factor.

Summary

The curve of growth has been determined for a number of iron lines by plotting solar equivalent breadths against intensities measured in an iron arc. The damping factor so found has been plotted against the quadratic Stark effect, and the relation between the two has led to the value of 14 dynes per cm.^{2} for the electron pressure of the Sun's reversing layer.

In conclusion, I wish to thank Dr. M. Minnaert for the use of his Utrecht step-wedge.

Commonwealth Solar Observatory,
Canberra :
1939 *June.*

PHOTOMETRY OF THE SOLAR D LINES

C. W. Allen, D.Sc.

Interest in the D lines is due to the fact that they are the only resonance lines at the red end of the spectrum strong enough to have zero central intensity on the basis of simplified Fraunhofer line theory. Although they have been measured several times * there has always been considerable doubt about the instrumental corrections. Particularly is this so for the central intensity. The present paper gives the results of a re-measurement using the corrections derived from a comparison with terrestrial water-vapour lines.

Observations.—Solar spectra of the D region were obtained with the Sun telescope and three-prism spectrograph of the Commonwealth Solar Observatory. Some plates were taken of both centre and limb of the Sun near midday. Others were exposed a few minutes after sunrise in order to ensure strong water-vapour (w-v) lines. All were calibrated with a step slit for which corrections for diffraction and scattered light had been applied. The spectra had a dispersion of 1·5 A. per mm., and were photometered with a Moll microphotometer having a linear magnification of 27. The three w-v lines at 5900·048, 5919·059 and 5919·647 A. could be used for obtaining the instrumental corrections since it will be shown that at rising Sun these have effectively zero central intensity. The measured central intensities of these w-v lines when plotted against their breadths define the correction curve to be used with the D lines.† The breadth was measured at a depth equal to 0·6 × the maximum depth, the fraction 0·6 having been selected in order to avoid distortion by blending, which is appreciable at the more usual 0·5. The correction curve so derived is shown in fig. 1, which contains also the D-line breadths for both centre and limb. It is seen that some extrapolation is necessary for correcting the D_2 lines. There is some doubt as to whether the difference in contour of the D lines and the w-v lines might have some effect on the correction. This is a difficulty that cannot be overcome without considerably higher resolution, but it is unlikely that the so-caused error in the central intensity would be greater than 2 per cent. of the continuous background.

For the limb spectra the spectrograph slit was set radially to the Sun's image and the microphotometer run at 2·1 mm. from the Sun's limb. With a 125 mm. solar image this represents a beam emerging at 75° to the normal at the Sun's surface.

* A. Unsöld, *Zs. f. Phys.*, **46**, 765, 1928 ; L. D'Azambuja, *Ann. d. l'Obs. Paris*, **8**, 67, 1930 ; S. A. Korff, *Ap. J.*, **76**, 291, 1932 ; C. W. Allen, *Mem. Comm. Sol. Obs. Canberra*, **1**, No. 5, 1934 ; *M.N.*, **96**, 508, 1936 ; J. Houtgast, *Zs. f. Astrophys.*, **16**, 43, 1938.

† C. W. Allen, *Mt. Wilson Contr.*, 567 ; *Ap. J.*, **85**, 165, 1937.

Water-Vapour Lines.—In order to be certain that the true central intensity of the w-v lines is zero, it is necessary to have some idea of the absorption line breadth. It has not been necessary to go through the labour of determining the curve of growth, since high dispersion spectra in this region made at Mount Wilson show that faint w-v lines are slightly sharper than the solar lines of comparable intensity. The breadths of the solar lines may be inferred from my earlier work*, where it can be seen that $W/(1-r_c)=83$

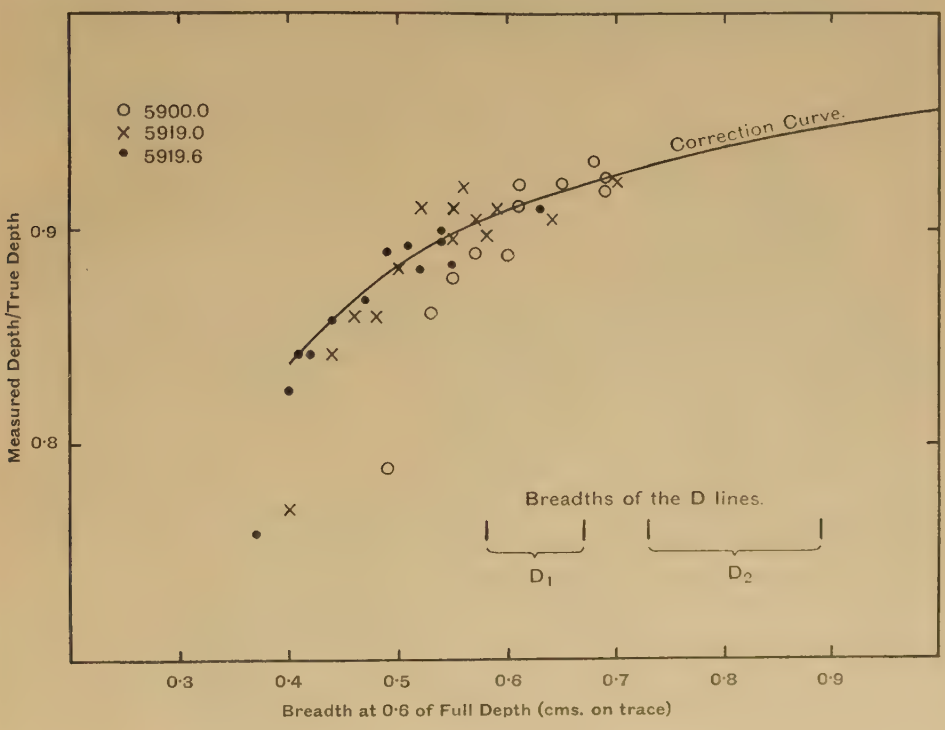

FIG. 1.

milli-A. for faint lines in this region. Here W is the equivalent breadth, r_c is the true central intensity, and the expression is a measure of line breadth. The w-v lines at Mount Wilson are certainly no broader than this, and allowing for the higher pressure on Mount Stromlo the factor $W(1-r_c)$ for w-v lines photographed here must be less than 93 m.-A. Assuming a contour of the form $1/\{(\lambda-\lambda_0)^2+\delta^2\}$ for the line absorption coefficient, we find the half breadth (2δ) less than $2/\pi \times 93$ or 59 m.-A. By plotting absorption line contours it may easily be verified that the w-v lines with an equivalent breadth greater than 230 m.-A. have zero central intensity. All w-v lines represented by points to the right of fig. 1 have an equivalent breadth greater than 230 m.-A. and consequently zero true central intensity. Although the w-v line at 5900·048 A. is complex, it is seen from fig. 1 to give the same correction curve.

* *Loc. cit.* (see fig. 1).

Results.—The results for the central intensities and equivalent breadths are given in Table I. They are in good agreement with my original observations *, but the central intensities are considerably smaller than the recent observations of Houtgast *, which, however, were not corrected for finite resolving power.

TABLE I

Central Intensities and Equivalent Breadths of the Sodium D Lines at the Centre and Limb of the Sun

Line	Central Intensity		Equivalent Breadth (W)
	Observed (r_m)	Corrected (r_c)	
	Per cent.	Per cent.	m.-A.
D_1 (5896 A.) centre	14·3	7·2	560
D_1 (5896 A.) limb	15·0	6·0	530
D_2 (5890 A.) centre	11·6	6·6	770
D_2 (5890 A.) limb	12·6	6·0	740

In plotting the observed contours blended lines have been eliminated as far as possible. From the observed contour it is possible to obtain the true contour by correcting for resolving power. Except at the centre of the line the correction is only small, and it is sufficiently accurate to assume

TABLE II

The Whole Breadth of the Sodium D Lines at Various Values of the Intensity r

r	D_1 (5896 A.)		D_2 (5890 A.)	
	Centre	Limb	Centre	Limb
	A.	A.	A.	A.
0·10	0·08	0·08	0·10	0·10
·20	·16	·14	·20	·19
·30	·23	·20	·30	·27
·40	·31	·26	·41	·35
·50	·39	·33	·52	·45
·60	·48	·43	·67	·59
·70	·62	·57	·87	·80
·80	·82	·80	1·15	1·12
·85	·98	·97	1·38	1·38
·90	1·23	1·23	1·70	1·75
·95	1·73	1·73	2·4	2·5
·98	2·5	2·5	3·5	3·7

that the correction is proportional to the curvature of the observed contour. The magnitude of the correction may be obtained for the whole line from the correction at the centre, and the corrected contour found in this way is given in Table II. The lines show evidence of asymmetry, being about 10 per cent. broader on the red side than the violet side. This asymmetry is in the opposite sense to that reported for other Fraunhofer lines by

* *Loc. cit.*

Redman.* It will be noticed that while the limb lines are narrower than those at the centre of the disc over the range $r = 20$ per cent. to $r = 80$ per cent., at the extreme wings they are approximately the same. A rather similar situation has been reported by Cherrington †, Minnaert ‡ and Houtgast § for several strong lines of *Fe*, *Ca*, Ca^+, *Na* and *Mg*. It might therefore be regarded as normal for the extreme wings of strong lines to have approximately the same intensity near the limb as at the centre of the Sun.

The Infra-red Ca^+ Lines.—An attempt to measure the contours of the infra-red Ca^+ lines by the same method was not very successful, as the plates had lost considerable sensitivity and were not of good quality. However, the values obtained for the central intensities of the lines in the centre of the Sun's disc are as follows :—

	8498 A.	8542 A.	8662 A.
	Per cent.	Per cent.	Per cent.
Corrected central intensity	35	26	28
Houtgast's central intensity	34	24	26

These results are in much better agreement with Houtgast's observations § than the D line central intensities. This is doubtless due to the greater width of the Ca^+ lines, which are consequently less affected by finite resolving power.

Summary

The D lines have been photometered at the centre and limb of the Sun. The corrections for finite resolving power were obtained from a comparison with strong terrestrial water-vapour lines.

The author is indebted to Dr. S. B. Nicholson of the Mount Wilson Observatory for a high dispersion spectrogram of the D region.

Commonwealth Solar Observatory,
Canberra :
1939 *August.*

* R. O. Redman, *M.N.*, **97**, 552, 1937.
† E. Cherrington, Jr., *L.O.B.*, **17**, No. 477, p. 161, 1935.
‡ M. Minnaert and J. Houtgast, *Zs. f. Astrophys.*, **12**, 81, 1936.
§ *Loc. cit.*

SOME THEOREMS FOR A STAR WITH VARIABLE POLYTROPIC INDEX

W. E. Candler, B.A.

1. Given a star of known mass and radius, whose density distribution is unknown except for some general restriction, it is possible to obtain limiting values for some of the other physical quantities connected with the star by comparing it with a model of the same mass and radius, whose density distribution is assumed to follow some definite law. The simplest such model is one of uniform density equal to the mean density of the given star. Alternatively, if the mass and central density of the given star are supposed known, then it can be compared with a model of the same mass and of uniform density equal to the central density of the given star. Thus it is known in particular that, if the mean density of the matter within a distance r from the centre of the star decreases outwards, then the central pressure, the mean value of gravity and the negative potential energy have values respectively greater for the given star than for the first of the above models and less than for the second. These and other related integral theorems, due to various authors, have been collected by Chandrasekhar.*

Our physical knowledge of actual stars is likely to provide the general restriction on the density distribution in the form of a restriction on the possible range of values of the polytropic index within a given star. In this case it is natural to use polytropic models in place of models of uniform density for the purpose of comparison. In particular we may enquire how far the properties of a star S, whose polytropic index n is known to vary between two limits n_1 and n_2, are intermediate between those of the polytropic models S_1, S_2 with constant indices n_1, n_2. In order to make the question definite, it is necessary to specify two of the quantities connected with the star, such as the mass and radius, or the central pressure and density, which are taken to have the same values for the models S_1, S_2 as for the star S. This question has recently been posed and discussed by Eddington †, and forms the subject of the present paper.

2. Before proceeding to the general question just outlined, it is desirable to set out the equations expressing the gravitational equilibrium of the star in a form which involves the polytropic index explicitly. The polytropic index n is defined by the equation

$$\frac{d}{dr}(\log P) = \left(1 + \frac{1}{n}\right)\frac{d}{dr}(\log \rho), \tag{1}$$

where P, ρ are the pressure and density at a distance r from the centre.

* *An Introduction to the Study of Stellar Structure*, 1939, chap. iii.

† *M.N.*, **99**, 4, 1938.

We introduce the dimensionless variables

$$X=\left(\frac{2\pi G}{3}\right)^{1/2}\frac{r\bar{\rho}(r)}{P^{1/2}}, \qquad Y=\left(\frac{2\pi G}{3}\right)^{1/2}\frac{r\rho}{P^{1/2}}, \tag{2}$$

where $\bar{\rho}(r)$ is the mean density of the matter within a distance r from the centre, and G is the constant of gravitation. The variable X is a positive quantity increasing from zero at the centre $r=0$ to infinity at the surface $P=0$; the variable Y is a positive quantity increasing initially from zero at the centre, and reaching a value at the surface which may be zero, finite or infinite according to the surface value of n.

If we differentiate X and Y with respect to r, we find, on using the hydrostatic equation

$$\frac{1}{\rho}\frac{dP}{dr}=-\frac{GM(r)}{r^2}, \tag{3}$$

where

$$M(r)=\tfrac{4}{3}\pi\bar{\rho}(r)r^3 \tag{4}$$

is the mass within a distance r from the centre, together with

$$\frac{dM(r)}{dr}=4\pi\rho r^2, \tag{5}$$

that X and Y satisfy the simultaneous differential equations of the first order

$$r\frac{dX}{dr}=(3+X^2)Y-2X, \tag{6}$$

$$r\frac{dY}{dr}=Y-\frac{n-1}{n+1}XY^2. \tag{7}$$

Dividing (7) by (6), we obtain the fundamental differential equation of the problem

$$\frac{dY}{dX}=\frac{Y-\frac{n-1}{n+1}XY^2}{(3+X^2)Y-2X}. \tag{8}$$

3. It is convenient now to regard X as the independent variable throughout a star, and the polytropic index n as a function of X. We can then obtain explicit expressions in terms of X and Y for the variables r, $M(r)$, $g(r)$, $\bar{\rho}(r)$, ρ and P, where

$$g(r)=\frac{GM(r)}{r^2} \tag{9}$$

is the value of gravity at a distance r from the centre. It will be found that these expressions only involve n through the function Y. We may choose as the constants of the problem either the central pressure P_c and the central density ρ_c, or the total mass M and the external radius R, and we shall give alternative expressions for these two cases.

We can rewrite (6) in the form

$$\frac{d(\log r)}{dX} = \frac{1}{(3+X^2)Y - 2X}, \tag{10}$$

and hence we have

$$\frac{d}{dX}\log\frac{r}{X} = -\frac{1}{X}\frac{(3+X^2)Y - 3X}{(3+X^2)Y - 2X}. \tag{11}$$

Now at the centre we see from the definition (2) of X that

$$\underset{X\to 0}{Lt}\,(r/X) = R_0, \tag{12}$$

where

$$R_0 = \left(\frac{3}{2\pi G}\right)^{1/2}\frac{P_c^{1/2}}{\rho_c} \tag{13}$$

is the radius of the model $n=0$ of uniform density with given P_c, ρ_c. Integrating (11), we obtain therefore

$$r = R_0 X \exp\left[-\int_0^X \frac{(3+X^2)Y - 3X}{(3+X^2)Y - 2X}\frac{dX}{X}\right], \tag{14}$$

while, in terms of the radius R, we have, from (10), the alternative expression

$$r = R\exp\left[-\int_X^\infty \frac{dX}{(3+X^2)Y - 2X}\right]. \tag{14'}$$

The expressions for the other variables follow in a similar manner. From (2), (4) and (5) we find that

$$\frac{d(\log M(r))}{d(\log r)} = \frac{3Y}{X}, \tag{15}$$

which, on combination with (10), gives

$$\frac{d(\log M(r))}{dX} = \frac{3Y}{X}\frac{1}{(3+X^2)Y - 2X}, \tag{16}$$

and hence

$$\frac{d}{dX}\log\frac{M(r)}{X^3} = -\frac{3}{X}\frac{(2+X^2)Y - 2X}{(3+X^2)Y - 2X}. \tag{17}$$

Now at the centre we see that

$$\underset{X\to 0}{Lt}\,(M(r)/X^3) = M_0, \tag{18}$$

where

$$M_0 = \left(\frac{6}{\pi G^3}\right)^{1/2}\frac{P_c^{3/2}}{\rho_c^2} \tag{19}$$

is the mass of the model $n=0$ of uniform density with given P_c, ρ_c. Integrating (17), we obtain therefore

$$M(r) = M_0 X^3 \exp\left[-3\int_0^X \frac{(2+X^2)Y - 2X}{(3+X^2)Y - 2X}\frac{dX}{X}\right], \tag{20}$$

while, in terms of the mass M, we have, from (16), the alternative expression

$$M(r) = M \exp\left[-3\int_X^\infty \frac{Y}{(3+X^2)Y-2X}\frac{dX}{X}\right]. \tag{20'}$$

The expressions for $g(r)$ and $\bar{\rho}(r)$ can be obtained immediately from (14) and (20) or from (14′) and (20′). We find that

$$g(r) = g_0 X \exp\left[-\int_0^X \frac{XY dX}{(3+X^2)Y-2X}\right], \tag{21}$$

or

$$g(r) = g \exp\left[\int_X^\infty \frac{2X-3Y}{(3+X^2)Y-2X}\frac{dX}{X}\right], \tag{21'}$$

where

$$g_0 = (\tfrac{8}{3}\pi G P_c)^{1/2} \tag{22}$$

is the surface value of gravity for the model $n=0$ with given P_c, and $g = GM/R^2$ is the surface value of gravity for a star with given M, R. Similarly for $\bar{\rho}(r)$ we find

$$\bar{\rho}(r) = \rho_c \exp\left[-3\int_0^X \frac{X-Y}{(3+X^2)Y-2X}\frac{dX}{X}\right], \tag{23}$$

or

$$\bar{\rho}(r) = \bar{\rho} \exp\left[3\int_X^\infty \frac{X-Y}{(3+X^2)Y-2X}\frac{dX}{X}\right], \tag{23'}$$

where $\bar{\rho}$ is the mean density of the star as a whole. We may note the following simple expression for the ratio of central to mean density for any star :—

$$\frac{\rho_c}{\bar{\rho}} = \exp\left[3\int_0^\infty \frac{X-Y}{(3+X^2)Y-2X}\frac{dX}{X}\right]. \tag{24}$$

The expressions for ρ, obtained from those for $\bar{\rho}(r)$ by introducing (2), are

$$\rho = \rho_c \frac{Y}{X} \exp\left[-3\int_0^X \frac{X-Y}{(3+X^2)Y-2X}\frac{dX}{X}\right], \tag{25}$$

or

$$\rho = \bar{\rho}\frac{Y}{X} \exp\left[3\int_X^\infty \frac{X-Y}{(3+X^2)Y-2X}\frac{dX}{X}\right]. \tag{25'}$$

Similarly, the expressions for P follow from those for $g(r)$, if we note the relation

$$P = \frac{3}{8\pi G}\frac{g^2(r)}{X^2}, \tag{26}$$

obtained from (2), (4) and (9). We find

$$P = P_c \exp\left[-2\int_0^X \frac{XY dX}{(3+X^2)Y-2X}\right], \tag{27}$$

or

$$P = \frac{P_0}{X^2} \exp\left[2\int_X^\infty \frac{2X-3Y}{(3+X^2)Y-2X}\frac{dX}{X}\right], \tag{27'}$$

where

$$P_0 = \frac{3GM^2}{8\pi R^4} \tag{28}$$

is the central pressure of the model $n=0$ of uniform density with given M, R. All the above expressions are merely formal consequences of the fact that the matter is in equilibrium, and make no other statement about the star.

4. We return now to a consideration of the solutions of the fundamental differential equation (8). We shall refer to the curve in the (X, Y) plane through the origin as the solution-curve for the star S. We are only concerned with that part of the curve which lies in the quadrant $X \geqslant 0$, $Y \geqslant 0$.

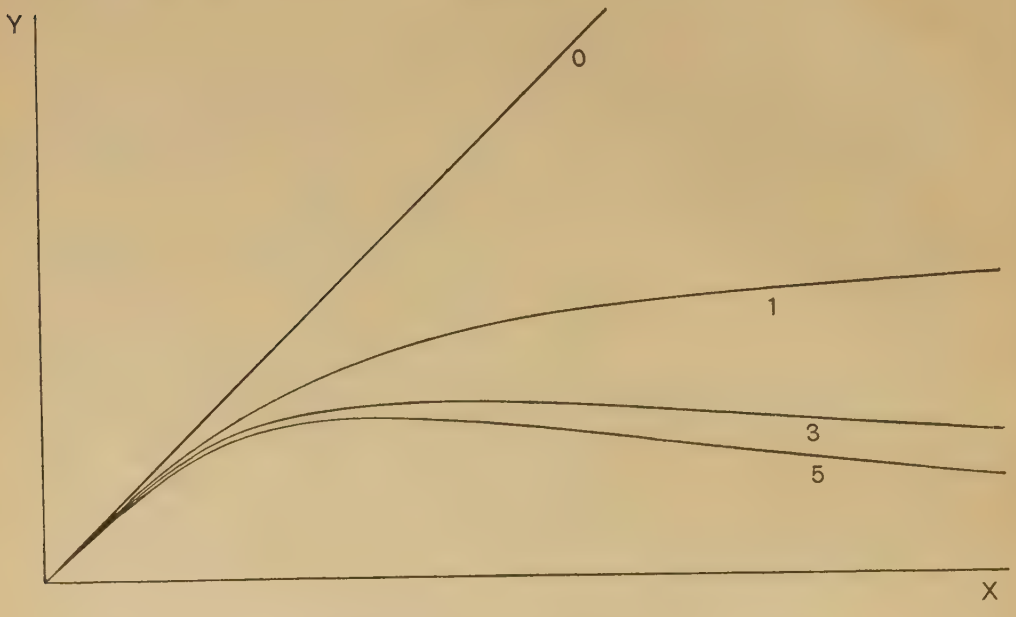

Fig. 1.—*Solution-curves for the polytropes* $n=0$, 1, 3 *and* 5.

The general arrangement of the solution-curves for the particular case of polytropic models, where n is constant, is shown in fig. 1, where the curves corresponding to the polytropes $n=0$, 1, 3 and 5 have been sketched. We may note that the equations to the curves for the polytropes $n=0$, 1 and 5 are

$$X - Y = 0, \tag{29}$$

$$XY + \sqrt{3}\,Y \cot(\sqrt{3}\,Y) - 1 = 0 \tag{30}$$

and

$$(3 + X^2)Y - 3X = 0 \tag{31}$$

respectively. For a polytrope with $n > 5$, the solution-curve will spiral round the singular point

$$X = \left[\frac{3(n+1)}{n-3}\right]^{1/2}, \qquad Y = \left[\frac{(n+1)(n-3)}{3(n-1)^2}\right]^{1/2}. \tag{32}$$

We consider now the properties of the general solution-curve corresponding to a star S, in which n is variable :

(i) The solution-curve is continuous, even if there are points in the star S at which n is discontinuous. This is clear from the definitions of X and Y, since P, ρ and $\bar{\rho}(r)$ are essentially continuous functions of r.

(ii) The solution-curve leaves the origin with unit slope. This follows immediately from the relation

$$Y/X = \rho/\bar{\rho}(r). \tag{33}$$

Near the origin we may expand Y as a power series in X in the form

$$Y = X - \frac{2}{5}\frac{n_c}{n_c+1}X^3 + O(X^4), \tag{34}$$

where n_c is the value of n at the centre. It follows that if the polytropic indices n, n' of two stars S, S' are such that $n_c < n_c'$, then the solution-curve for S lies initially above the solution-curve for S'.

(iii) If $\bar{\rho}(r)$ decreases outwards at all points within the star S, then the solution-curve for S lies entirely below the solution-curve for the model $n=0$ of uniform density. For the condition $d\bar{\rho}(r)/dr < 0$ is equivalent to $\bar{\rho}(r) > \rho$, or, from (33), to $X > Y$, and the result follows from (29).

(iv) If $-1 < n < 5$ at all points within the star S, then the solution-curve for S lies entirely above the solution-curve for the polytrope $n=5$. We have seen in (ii) above that this holds good for small values of X. In order to prove the result for all X, we notice that the solution-curve (31) for the polytrope $n=5$ lies entirely above the curve

$$(3+X^2)Y - 2X = 0, \tag{35}$$

which is the locus of points at which the solution-curves have vertical tangents. Hence, so long as the solution-curve for S lies above (31), its tangent cannot become vertical, and if it crosses (31) it must do so from above to below in such a way that at the point of intersection $(dY/dX)_n < (dY/dX)_5$, where the suffix n refers to the star S and the suffix 5 to the polytrope $n=5$. This, however, is impossible from the differential equation (8), the denominator on the right-hand side of that equation being positive, and the result is accordingly proved.

(v) If $-1 < n < 5$ at all points within the star S, then X, X/r and XY increase monotonically with r. We have just seen that, if $-1 < n < 5$ everywhere, then $Y > 3X/(3+X^2)$, so that these results follow from (6) and the equations obtained from (6) and (7), namely

$$r^2\frac{d}{dr}\left(\frac{X}{r}\right) = (3+X^2)Y - 3X \tag{36}$$

and

$$r\frac{d(XY)}{dr} = \left(3+\frac{2}{n+1}X^2\right)Y\left(Y-\frac{3X}{3+X^2}\right) + \left(6+\frac{5-n}{n-1}X^2\right)\frac{XY}{3+X^2}. \tag{37}$$

Since X increases monotonically with r, we may regard r, and consequently

all other variables connected with the star, as single-valued functions of X. Further, if we substitute the definition of X from (2), we may express the result that X/r increases monotonically with r in the form :

If $-1 < n < 5$ at all points within a star, then the function $P/\bar{\rho}^2(r)$ decreases outwards.

(vi) If the polytropic indices n, n' of two stars S, S' are such that $-1 < n < n' \leqslant 5$ at all "corresponding" points (*i.e.* points with the same value of X), then the solution-curve for S lies entirely above the solution-curve for S'. We have seen in (ii) above that this holds good for small values of X, and in (v) that the tangents to the solution-curves cannot become vertical. Hence if the solution-curve for S crosses the solution-curve for S', it must do so from above to below in such a way that at the point of intersection $(dY/dX)_n < (dY/dX)_{n'}$, which is impossible from the differential equation (8). It follows that if n varies in any manner between a lower limit $n_1(> -1)$ and an upper limit $n_2(\leqslant 5)$, then the solution-curve for S lies entirely below the solution-curve for the polytrope $n = n_1$, and above the solution-curve for the polytrope $n = n_2$. It is this property of the solution-curves that gives them their importance.

5. We proceed now to comparisons of the type outlined at the beginning of the paper. We consider two stars S, S' whose polytropic indices n, n' are variable quantities such that $n < n'$ at all "corresponding" points (*i.e.* points with the same value of X), and compare the values of other variables connected with the stars at such corresponding points. The choice of X as the variable which determines what points in the two stars are supposed to correspond is arbitrary, and similar results could be obtained by a different choice. It has the advantage of simplicity, however, in that X has the same range of values, from zero to infinity, for any star, and the analysis is the same whether we are comparing two stars with the same central pressure and density or with the same mass and radius. Since we are ultimately concerned with the comparison of integral properties of the two stars, the choice is not of great importance.

Theorem I.—If the polytropic indices n, n' of two stars S, S' with the same central pressure and density vary in any manner between centre and surface such that $-1 < n < n' \leqslant 5$ at all corresponding points, then the values of r and $M(r)$ are less, and the values of $g(r)$, $\bar{\rho}(r)$, ρ and P are greater at any point in S than at the corresponding point in S'.

Proof.—We denote the values of r, $M(r)$ . . . in the two stars S and S' by $r(X)$, $M(X)$ . . . and $r'(X)$, $M'(X)$ . . . to emphasize the fact that we are comparing the values of these variables at two points with the same value of X. We have already seen that if n, $n' \leqslant 5$, then these variables are single-valued functions of X. Expressions for the six variables $r(X)$, $M(X)$, $g(X)$, $\bar{\rho}(X)$, $\rho(X)$ and $P(X)$ in terms of P_c and ρ_c have been given in equations (14), (20), (21), (23), (25) and (27). If we divide these equations by the corresponding expressions for $r'(X)$, $M'(X)$ . . ., we obtain

$$\frac{r(X)}{r'(X)} = \exp\left[-\int_0^X \frac{(3+X^2)(Y-Y')dX}{\{(3+X^2)Y - 2X\}\{(3+X^2)Y' - 2X\}} \right], \qquad (38)$$

$$\frac{M(X)}{M'(X)} = \exp\left[-6\int_0^X \frac{(Y-Y')dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y'-2X\}}\right], \tag{39}$$

$$\frac{g(X)}{g'(X)} = \exp\left[2\int_0^X \frac{X^2(Y-Y')dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y'-2X\}}\right], \tag{40}$$

$$\frac{\bar{\rho}(X)}{\bar{\rho}'(X)} = \exp\left[3\int_0^X \frac{(1+X^2)(Y-Y')dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y'-2X\}}\right], \tag{41}$$

$$\frac{\rho(X)}{\rho'(X)} = \frac{Y}{Y'}\exp\left[3\int_0^X \frac{(1+X^2)(Y-Y')dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y'-2X\}}\right] \tag{42}$$

and

$$\frac{P(X)}{P'(X)} = \exp\left[4\int_0^X \frac{X^2(Y-Y')dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y'-2X\}}\right]. \tag{43}$$

But we have seen that if $-1 < n(X) < n'(X) \leqslant 5$, then $Y > Y'$, $(3+X^2)Y-2X > 0$ and $(3+X^2)Y'-2X > 0$. Thus the integrands in (38) to (43) are positive for all values of X, while the factor Y/Y' in (42) is greater than unity, and the statements in the theorem are proved. The results for $g(X)$ and $\bar{\rho}(X)$ are special cases of the result that

$$M(X)/r^{\nu}(X) > M'(X)/r'^{\nu}(X)$$

for $\nu \geqslant 2$, which can be seen from (38) and (39).

Corollary 1.—Under the conditions of Theorem I, the mass and radius are less, and the mean density and surface value of gravity are greater for S than for S'.

Corollary 2.—If the polytropic index n of a star S varies in any manner between a lower limit $n_1(>-1)$ and an upper limit $n_2(\leqslant 5)$, then the mass and radius of S are greater than those of S_1' and less than those of S_2', while the mean density and the surface value of gravity of S are less than those of S_1' and greater than those of S_2', where S_1', S_2' are the polytropic models, with constant indices n_1, n_2, having the same central pressure and density as S.

6. We have been concerned in the last section with the comparison of stars and models with the same central pressure and density. It is of greater physical interest to compare stars and models with the same mass and radius, for which similar results hold good.

Theorem II.—If the polytropic indices n, n^* of two stars S, S^* with the same mass and radius vary in any manner between centre and surface such that $-1 < n < n^* \leqslant 5$ at all corresponding points, then the values of $g(r)$, $\bar{\rho}(r)$ and P are less, and the values of r, $M(r)$ and $P/\bar{\rho}^{\nu}$ for $\nu \geqslant 4/3$ are greater at any point in S than at the corresponding point in S^*.

Proof.—Denoting the values of r, $M(r)$ . . . at corresponding points in S and S^* by $r(X)$, $M(X)$ . . . and $r^*(X)$, $M^*(X)$ . . ., and dividing the expressions (21′), (23′), (27′), (14′) and (20′) by the corresponding expressions for S^*, we obtain

$$\frac{g(X)}{g^*(X)} = \exp\left[-2\int_X^{\infty} \frac{X^2(Y-Y^*)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^*-2X\}}\right], \tag{44}$$

$$\frac{\bar{\rho}(X)}{\bar{\rho}^*(X)} = \exp\left[-3\int_X^\infty \frac{(1+X^2)(Y-Y^*)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^*-2X\}}\right], \tag{45}$$

$$\frac{P(X)}{P^*(X)} = \exp\left[-4\int_X^\infty \frac{X^2(Y-Y^*)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^*-2X\}}\right], \tag{46}$$

$$\frac{r(X)}{r^*(X)} = \exp\left[\int_X^\infty \frac{(3+X^2)(Y-Y^*)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^*-2X\}}\right] \tag{47}$$

and

$$\frac{M(X)}{M^*(X)} = \exp\left[6\int_X^\infty \frac{(Y-Y^*)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^*-2X\}}\right], \tag{48}$$

while combination of (45) and (46) gives

$$\frac{P(X)/\{\bar{\rho}(X)\}^\nu}{P^*(X)/\{\bar{\rho}^*(X)\}^\nu} = \exp\left[\int_X^\infty \frac{\{3\nu+(3\nu-4)X^2\}(Y-Y^*)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^*-2X\}}\right]. \tag{49}$$

The integrands are again positive for all values of X in (44) to (48) and in (49) for $\nu \geqslant 4/3$, and the theorem is proved. No such result holds for the density ρ; it is easily seen that $\rho(X) < \rho^*(X)$ for small values of X, while $\rho(X) > \rho^*(X)$ for large values of X.

Corollary 1.—Under the conditions of Theorem II, the central pressure P_c and the central density ρ_c are less, while the value of P_c/ρ_c^{ν} for $\nu \geqslant 4/3$ is greater for S than for S^*.

Corollary 2.—If the polytropic index n of a star S varies in any manner between a lower limit $n_1(>-1)$ and an upper limit $n_2(\leqslant 5)$, then the central pressure P_c and the central density ρ_c of S are greater than those of S_1^* and less than those of S_2^*, while the value of P_c/ρ_c^{ν} for $\nu \geqslant 4/3$ is less than for S_1^* and greater than for S_2^*, where S_1^*, S_2^* are the polytropic models, with constant indices n_1, n_2, having the same mass and radius as S.

A result similar to this last corollary has been proved by Eddington * for the negative potential energy of a star.

7. We may use the result just obtained for the value of $P_c/\rho_c^{4/3}$ to extend two theorems due to Chandrasekhar.†

Theorem III.—If the polytropic index n of a wholly gaseous star varies in any manner between a lower limit $n_1(>-1)$ *and an upper limit* $n_2(\leqslant 5)$, *then the ratio* $(1-\beta_c)$ *of the radiation pressure to the total pressure at the centre of the star satisfies the inequalities*

$$1-\beta_c(n_2) < 1-\beta_c < 1-\beta_c(n_1), \tag{50}$$

where $\beta_c(n_1)$, $\beta_c(n_2)$ *are the solutions of the quartic equations*

$$\frac{1-\beta_c(n_i)}{\beta_c^4(n_i)} = D(n_i)\frac{\pi a G^3 H^4}{18k^4}\mu_c^4 M^2, \qquad (i=1,2) \tag{51}$$

$D(n_i)$ *being a constant depending only on* n_i, *and the other symbols having their usual meaning.*

* *M.N.*, **91**, 440, 1931.
† *Ap. J.*, **85**, 372, 1937; **86**, 78, 1937: **87**, 535, 1938.

Proof.—For a polytropic model with constant index n_i, we find from (23′) and (27′) that

$$\frac{P_c}{\rho_c^{4/3}} = C(n_i)\frac{G}{2}\left(\frac{4\pi}{3}\right)^{1/3} M^{2/3}, \tag{52}$$

where

$$C(n_i) = \underset{X\to 0}{Lt}\,\frac{1}{X^2}\exp\left[-2\int_X^\infty \frac{Y}{(3+X^2)Y-2X}\frac{dX}{X}\right]. \tag{53}$$

Hence, from the second corollary to Theorem II, we have

$$C(n_2)\frac{G}{2}\left(\frac{4\pi}{3}\right)^{1/3} M^{2/3} < \frac{P_c}{\rho_c^{4/3}} < C(n_1)\frac{G}{2}\left(\frac{4\pi}{3}\right)^{1/3} M^{2/3}. \tag{54}$$

We may note in particular for the polytropes $n=0$, 1 and 5, that $C(0)=1$, $C(1)=(3/\pi^2)^{1/3}$ and $C(5)=1/3$, as may be verified by substituting (29), (30) and (31) in (53).

Now in a wholly gaseous star the gas pressure p_g and the radiation pressure p_r are given by

$$p_g = \beta P = \frac{k}{\mu H}\rho T \tag{55}$$

and

$$p_r = (1-\beta)P = \tfrac{1}{3}aT^4, \tag{56}$$

where T is the temperature at any point and the other symbols have their usual meaning. Eliminating T between (55) and (56), and substituting the values at the centre, we obtain

$$\frac{1-\beta_c}{\beta_c^4} = \frac{a}{3}\left(\frac{\mu_c H}{k}\right)^4 \frac{P_c^3}{\rho_c^4}, \tag{57}$$

which, with (54), gives

$$D(n_2)\frac{\pi a G^3 H^4}{18k^4}\mu_c^4 M^2 < \frac{1-\beta_c}{\beta_c^4} < D(n_1)\frac{\pi a G^3 H^4}{18k^4}\mu_c^4 M^2, \tag{58}$$

where $D(n_i)=C^3(n_i)$. But $(1-\beta)/\beta^4$ is a monotonic increasing function of $(1-\beta)$, and the theorem is proved.

Table I gives the values of the factor $D(n_i)$ for $n_i = 0(0{\cdot}5)5$, together with the values of $(1-\beta_c(n_i))$ obtained from (51) for these values of n_i and various values of $\mu_c^2(M/M_\odot)$.

Thus for a star with mass $5M_\odot$, whose polytropic index is known not to range outside the limits 1·5 and 3·5, we see from the table that $(1-\beta_c)$ lies between 0·05 and 0·11, if we assume that the mean molecular weight at the centre is unity.

Theorem IV.—Under the conditions of Theorem III, the average of $\kappa\eta$ *throughout the star* (*taken with respect to the total pressure*) *satisfies the inequalities*

$$\frac{4\pi cGM(1-\beta_c(n_2))}{L} < \overline{\kappa\eta} < \frac{4\pi cGM(1-\beta_c(n_1))}{L}. \tag{59}$$

TABLE I

Values of $(1-\beta_c(n_i))$

n_i	$D(n_i)$	$\mu_e^2(M/M_\odot)$							
		0·5	1	2	5	10	20	50	100
0	1·0000	0·0081	0·0296	0·091	0·256	0·41	0·55	0·70	0·78
0·5	0·4958	·0041	·0155	·053	·184	·33	·48	·65	·74
1	·3040	·0025	·0098	·035	·139	·28	·43	·61	·71
1·5	·2085	·0017	·0068	·025	·109	·24	·39	·57	·68
2	·1529	·0013	·0050	·019	·088	·20	·35	·55	·66
2·5	·1170	·0010	·0038	·015	·072	·18	·32	·52	·64
3	·0921	·0008	·0030	·012	·060	·16	·30	·50	·62
3·5	·0737	·0006	·0024	·009	·050	·14	·27	·47	·60
4	·0594	·0005	·0020	·008	·042	·12	·25	·45	·59
4·5	·0478	·0004	·0016	·006	·035	·10	·23	·43	·57
5	0·0370	0·0003	0·0012	0·005	0·028	0·09	0·20	0·40	0·54

Proof.—In this theorem κ denotes the opacity coefficient, and η the ratio

$$\eta = \frac{L(r)/L}{M(r)/M}, \tag{60}$$

where $L(r)$ is the net amount of energy crossing a spherical surface of radius r, and L is the luminosity of the star. Then, by a theorem due to Strömgren *,

$$\overline{\kappa\eta} = \frac{4\pi cGM(1-\beta_c)}{L}, \tag{61}$$

where

$$\overline{\kappa\eta} = \frac{1}{P_c}\int_0^{P_c} \kappa\eta dP. \tag{62}$$

Combining (50) and (61), we obtain the result required.

Substituting numerical values, the inequality (59) becomes

$$1{\cdot}32\times 10^4\frac{M/M_\odot}{L/L_\odot}(1-\beta_c(n_2)) < \overline{\kappa\eta} < 1{\cdot}32\times 10^4\frac{M/M_\odot}{L/L_\odot}(1-\beta_c(n_1)). \tag{63}$$

Thus for a star with the mass and luminosity of the Sun, whose polytropic index is known not to range outside the limits 1·5 and 3·5, we find from Table I that $\overline{\kappa\eta}$ lies between 32 and 89 gm.$^{-1}$ cm.^{2}, on the assumption again that $\mu_e = 1$. If, further, we assume that the rate of liberation of energy decreases outwards, so that $\eta > 1$ everywhere, then this sets an upper limit to the mean value (taken with respect to the total pressure) of the opacity coefficient.

Summary

It is shown that if the polytropic index of a star varies between two limits n_1 and n_2, then certain of the properties of the star are intermediate between those of the polytropic models with constant indices n_1, n_2.

* *Handbuch der Astrophysik*, **7**, 159, 1936.

THE QUANTAL THEORY OF CONTINUOUS ABSORPTION OF RADIATION BY VARIOUS ATOMS IN THEIR GROUND STATES

I. The Atoms from Boron to Neon

D. R. Bates, M.Sc., University College, London

(Communicated by H. R. Hulme, Ph.D.)

Introduction.—In a number of astrophysical problems, such as arise in the detailed study of stellar atmospheres, it is necessary to have a knowledge of the cross-section associated with the absorption of a photon (and the subsequent ejection of an electron) by various atoms. For hydrogen and a few other isolated elements this cross-section (generally known as the absorption coefficient) has been properly calculated over a range of frequencies * ; unfortunately, however, for most elements there is only available the very dubious information derived from crude atomic models. Such models, though probably giving correctly the order of magnitude of the absorption coefficient, cannot be relied upon to give the correct variation with frequency—a matter of considerable importance to the astrophysicist. The present paper gives an account of a preliminary investigation of the continuous absorption coefficients of the normal (*i.e.* unexcited) atoms from boron to neon by a method which can be expected to give, not only the magnitude of the coefficient, but also the correct variation with frequency.

It is hoped in later papers to extend the investigation, with possibly increased accuracy, to other groups where both the coefficients themselves and the frequency variations may be considerably different. It is also hoped to study such excited and ionised atoms as are likely to be of most interest.

The Absorption Coefficients

1. Quantal perturbation theory shows that for an elementary atomic system containing but a single electron the absorption coefficient is given by the relation

$$a = \frac{32\pi^4 m^2 e^2}{3h^3 c} \nu v \left| \int \chi_i^* \mathbf{r} \chi_f d\tau \right|^2. \qquad (1)$$

Here m is the electronic mass, e the electronic charge, h Planck's constant, and c the velocity of light. ν is the frequency of the absorbed photon and v the velocity of the ejected electron. χ_i is the initial discrete wave function

* Hargreaves, *Proc. Camb. Phil. Soc.*, **25**, 75, 1929 ; Trumpy, *Zeits. f. Physik.*, **71**, 720, 1931 ; Phillips, *Phys. Rev.*, **39**, 905, 1932 ; Vinti, *Phys. Rev.*, **44**, 524, 1933 ; Bates, Buckingham, Massey and Unwin, *Proc. Roy. Soc.*, A, **170**, 322, 1939. For hydrogen, Sugiura, *Journ. de Phys.*, **8**, 113, 1927.

and χ_f the final continuous wave function. (As usual χ_i is normalised to unit density, and χ_f is normalised to have the asymptotic form of a modified plane wave of unit amplitude together with the corresponding scattered wave.) Now the true atomic system is a complex one. It includes in addition to the active electron a number of others which are disturbed to a slight extent, as a result of the photo-ionisation. The effect of this is that the probability of the process is reduced by the factor *,

$$|\int \phi_i^* \phi_f d\tau|^2, \qquad (2)$$

ϕ_i and ϕ_f being respectively the initial and final wave functions (both normalised to unit density) of the passive electrons. This correction term in practice differs but very little from unity. It is of much more importance that *any* electron of the atom can be ejected. In principle this renders it necessary to deal with what may be regarded as the summation of a series of partial absorption coefficients,

$$\alpha_{\text{Total}} = \sum_{\text{All electrons}}^{s} \alpha_s \ . \qquad (3)$$

In fact, however, unless the energy of the incident radiation is very high, only the electrons in the loosely bound outermost shell need be taken into account: and if to a first approximation these are taken as identical then it follows simply that

$$\alpha_{\text{Total}} = N\alpha, \qquad (4)$$

where N is the number in the shell and α is the partial absorption coefficient (as given by the ordinary formula) associated with each.

The actual calculation of the absorption coefficient will now be described. For any particular atom it involves essentially the determination of the wave functions and the evaluation of the integrals concerned. As is customary in such work it is convenient to employ atomic units throughout.

2.1. *The Initial Discrete Wave Function* is of the form,

$$\chi_i = S_i(\theta, \phi) R_i(r). \qquad (5)$$

Since for boron, carbon, nitrogen, oxygen, fluorine and neon the azimuthal quantum number is 1 (the electron being ejected from the $2p$ shell), we may take $S_i(\theta, \phi)$, the angular component of the wave function, as one of

$$\sin\theta \sin\phi,$$
$$\cos\theta,$$
$$\sin\theta \cos\phi.$$

$R_i(r)$ the radial wave function can be got with considerable accuracy from the tables (based on self-consistent field calculations) that have been published by various workers (Hartree and Black †; Torrence ‡; Brown, Bartlett and Dunn §) for the elements in which we are interested.

* It is assumed here that the wave functions for the individual electrons are completely separable and orthogonal.

† *Proc. Roy. Soc.*, A, **139**, 311, 1933.

‡ *Phys. Rev.*, **46**, 388, 1934.

§ *Phys. Rev.*, **44**, 214, 296, 1933.

2.2. *The Final Continuous Wave Function* may be expanded in the series

$$\chi_f = \sum_{t=0}^{\infty} (2t+1) P_t(\cos\theta) i^t e^{i\eta} L_t(r). \qquad (6)$$

Here $P_t(\cos\theta)$ is the tth Legendre Coefficient, and, if we write k for the momentum of the free electron and $U(r)$ for its potential energy in the field of the ion, then, neglecting the effects of polarisation and exchange, $L_t(r)$ is that solution of the differential equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dL_t}{dr}\right)+\left(k^2-U-\frac{t(t+1)}{r^2}\right)L_t=0, \qquad (7)$$

which is everywhere finite, single valued and continuous and which, for large r, tends to

$$(kr)^{-1}\sin\left(kr-\tfrac{1}{2}t\pi+\eta_t-\frac{1}{k}\log 2kr\right).$$

Using for U the values of the undisturbed ionic field given by the self-consistent field method, the functions L_t may be determined by numerical or mechanical solution of the differential equations (7). Such calculations have previously been carried out for oxygen*, and from the experience gained in this case it is possible to avoid repeating the work in detail for the other atoms. For absorption by electrons in p-states, only L_0 and L_2 are required. It is found from the oxygen results that the contribution from L_2 is much greater than from L_0 and, further, that the atomic field of the oxygen core does not appreciably modify the form of L_2 from that, L_2^C, which it would have for a pure Coulomb field (*i.e.* if U in (7) $= 1/r$). For lighter atoms L_2 will differ even less from L_2^C and the increased field even for neon is unlikely to be great enough to produce much extra modification. Hence in calculating absorption coefficients for the atoms concerned the same continuous functions may be used as in the oxygen calculations, the discrete wave functions alone varying markedly from atom to atom. This approximation was therefore used throughout. It is unlikely to lead to any serious error for the light atoms we have considered, but it cannot be expected to hold for heavier atoms in which L_2 may differ markedly from L_2^C.

3. *Discussion of Results.*—The integrals occurring in (1) were evaluated numerically using the above discussed forms for χ_i and χ_f, and the final results of the calculation are illustrated in fig. 1.

It will be noticed that, as we pass from boron to neon, the absorption coefficient at the limit steadily decreases and the shape of the curve illustrating the variation of absorption coefficient with frequency changes progressively in form. The marked departure of the curves from those given by the ν^3 law can be understood in terms of the form of the wave function of the active electron. The integral which appears in the formula (1) for the absorption coefficient is determined mainly by the amount of overlap between the functions r^2R_i and rL_2, L_0 being unimportant. For all the atoms r^2R_i rises from zero at $r=0$ to a maximum value at $r=r_M$ (say) and then falls of

* Bates, Buckingham, Massey and Unwin, *Proc. Roy. Soc.*, A, **170**, 322, 1939.

exponentially. As we proceed from boron to neon R_i becomes more and more compact, r_M decreasing steadily from $3{\cdot}6\,a_0$ for boron to $1{\cdot}2\,a_0$ for neon. The function rL_2 for a given electron energy in the energy range concerned varies very slightly from atom to atom. For electrons of very small energy it rises gradually from zero to a maximum at $r = r_N (\approx 6a_0)$ and then falls steadily to zero at $r = r_P (\approx 10a_0)$ after which it oscillates with a certain wave-length. As the electron energy increases, r_N, r_P and the wave-length of the oscillations all decrease. The value of the integral depends

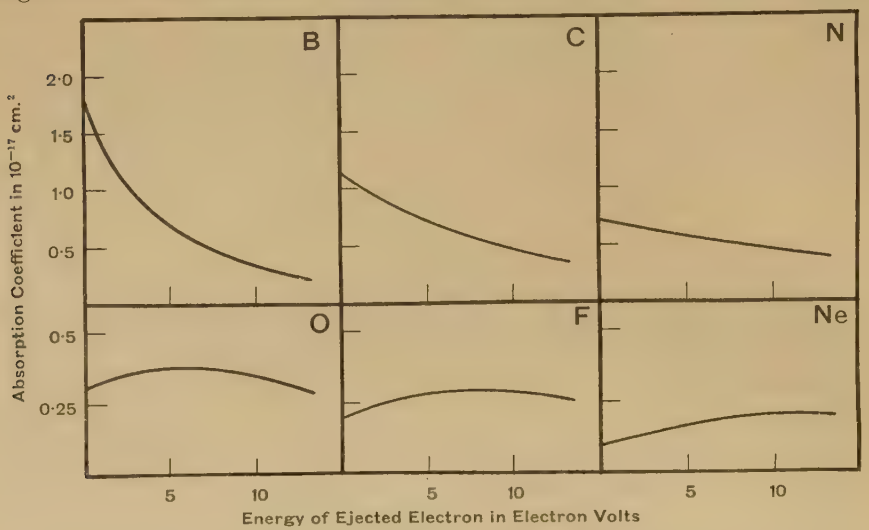

FIG. 1.—*Illustrating the continuous absorption coefficients (per 2p electron) of the normal atoms from boron to neon as functions of the energy of the ejected electron. In terms of the wave number of the absorbed radiation the ranges covered are approximately as follows, in* 10^4 *cm.*$^{-1}$ *:—*

Boron	$6{\cdot}7 \rightarrow 17{\cdot}7$	Oxygen	$10{\cdot}9 \rightarrow 21{\cdot}9$
Carbon	$9{\cdot}1 \rightarrow 20{\cdot}1$	Fluorine	$13{\cdot}6 \rightarrow 24{\cdot}6$
Nitrogen	$11{\cdot}7 \rightarrow 22{\cdot}7$	Neon	$17{\cdot}4 \rightarrow 28{\cdot}4$

on the difference between the overlap of r^2R_i with the positive part of rL_2 between $r = 0$ and r_P and the negative first oscillation of rL_2 beyond $r = r_P$. In all cases r_M is less than r_N for low energy electrons, so the overlap with positive rL_2 increases with electron energy until r_M and r_N practically coincide. At the same time the negative oscillation also becomes more and more important. However, for the more compact atoms such as neon the first effect is more important initially than the second, giving the initial rise of α with increasing ν. For boron, carbon and nitrogen, on the other hand, the increase of the negative oscillations with frequency overshadows that of the positive from the limit outwards, giving the steady decrease of α with ν.

In conclusion it may be pointed out that absorption coefficients are frequently estimated from analytical expressions derived on the assumption that the wave functions for the discrete and continuous states can be described as hydrogen-like—the effective nuclear charge *for both* being taken as some semi-arbitrary constant Z. It must be emphasised that for normal atoms no reliance can be placed on such calculations as the most appropriate Z, for

the discrete state may be very different from that for the continuous state.* Further, it is incorrect to attempt to use an average value; such a procedure must always give a variation of absorption coefficient with frequency, following the same form as for atomic hydrogen.

For atoms, however, in which an electron is excited to a state of higher total quantum number than its ground state these objections are less important, and it will usually be quite a good approximation to suppose the absorption by such an atom to be the same as that for a hydrogen atom in which the electron is excited to a state of the same quantum numbers as in the atom concerned. Detailed consideration has been given to this point in a previous paper.†

It is of interest to notice also that, although exchange and polarisation effects have been neglected in determining the wave function χ_f for the ejected electron, this is unlikely to introduce any considerable error. This follows from the fact that the main contribution to the absorption coefficient in all cases comes from the component L_2 of χ_f. We have made use already of the result that this component differs only slightly from L_2^C, the corresponding function for a Coulomb field of charge unity. This means that L_2 is practically independent of the specific atomic field owing to the comparatively great importance of the centrifugal force potential $6/r^2$ and the asymptotic Coulomb potential $1/r$. Exchange and polarisation effects can be expected to be of the same order as any specific atomic field effects, and are therefore probably small for the atoms concerned. For heavier atoms, however, for which L_2 and L_2^C are markedly different, this may not be the case.

Summary

The coefficients for the continuous absorption of radiation by boron, carbon, nitrogen, oxygen, fluorine and neon atoms *in their ground states* are calculated, over a wide frequency range, by quantal methods. The investigation shows that, although the order of magnitude of the coefficients is similar to that indicated by the various rough approximations that have been employed, the variation of the absorption with frequency is in many cases completely different. This arises from the fact that it is a bad approximation to represent the wave functions of the initial and final states of the electron involved in the absorption process as hydrogen-like wave functions with the same effective nuclear charge. For the atoms concerned the best simple approximation is to take this effective nuclear charge, as given by Slater's rules, for the bound wave function, but as unity for the wave function representing the ejected electron.

I would like to take this opportunity of thanking Professor Massey for suggesting the problem and discussing the various points arising in it. I desire also to acknowledge a grant made available to me by the Ministry of Education for Northern Ireland.

* For the elements here considered the best simple approximation is to take Slater's value of Z for the discrete state; and to take Z equal to one for the continuous state.

† Bates, Buckingham, Massey and Unwin, *Proc. Roy. Soc.*, A, **170**, 322, 1939.

THE CONSTANTS OF THE STAR-STREAMS FROM THE CAPE PHOTOGRAPHIC PROPER MOTIONS OF 18,323 STARS

W. M. Smart, M.A., D.Sc., and T. R. Tannahill, M.A., B.Sc.

1. The proper motion material discussed in the present paper is to be found in a recent Cape publication *, which contains also a discussion of the methods of reduction and kindred matters by Dr. J. Jackson, H.M. Astronomer at the Cape, to whom, we may add, we are indebted for his interest in this investigation. The photographic zone lies between the limits $-40°$ and $-52°$ in declination, and altogether the survey extends over 1512 photographic areas.

The proper motions have been derived from three series of photographs, the first obtained between 1892 and 1896, the second between 1897 and 1905 (except in the case of a few areas) and the third between 1923 and 1928, the photographs of the last series being taken through the glass. For 1223 of the areas the first and third series, with an average interval of 34 years, were measured for proper motions ; for the remaining areas the second and third series, with an average interval of 24 years, were used.

Images of all stars down to magnitude 9·0 on the C.P.D. scale were measured, together with such fainter stars (10 for each area and of mean photographic magnitude 11·9) as were required as reference stars. The relative proper motions obtained in the usual way were transformed into absolute proper motions by applying corrections, depending on the parallactic motions of the reference stars and on galactic rotation, based on data given by Van Rhijn and Bok.†

The probable errors, in each co-ordinate, of the proper motions are stated by Jackson to be $\pm 0''{\cdot}008$, $\pm 0''{\cdot}0055$ and $\pm 0''{\cdot}0044$ for one, two and three determinations respectively.

The Catalogue also gives the spectral types of the individual stars.

2. In the Cape publication Jackson has given an analysis of the proper motions on the ellipsoidal hypothesis, using Schwarzschild's automatic method.‡ For this purpose he grouped the stars in regions bounded by the parallels of $-40°$ and $-52°$ and by meridians one hour apart. The richness and homogeneity of the Cape material make it desirable to analyse the proper motions on the alternative two-streams theory, and the present paper is an investigation on these lines. With a view to a future analysis, according to spectral type, it was considered expedient to increase the size of the regions so as to be able to deal with larger populations of stars. Accordingly, we have subdivided the zone into 16 equal regions by meridians $1\frac{1}{2}$ hours apart. The right ascensions of the centres of these regions are

* *Proper Motions of Stars in the Zone Catalogue of 20,843 Stars*, 1936.

† *Groningen Publ.*, No. 45, 1931.

‡ *Göttinger Nach.*, 171, 1908 ; W. M. Smart, *Stellar Dynamics*, 161, 1938.

given in Table I, the declinations of the geometrical centres being $-45\frac{3}{4}°$. The table also contains details as to the numbers* of stars of all assigned spectral types in our regions, the galactic co-ordinates of the centres and the weights assigned in the subsequent analysis.

TABLE I.

Region	R.A.	No. of Stars	Gal. Long.	Gal. Lat.	Weight
	h m		°	°	
I	0 45	638	267·1	−72·2	2
II	2 15	564	229·9	−64·2	2
III	3 45	706	218·9	−49·7	2
IV	5 15	819	218·3	−34·0	2
V	6 45	1112	222·6	−18·8	2
VI	8 15	2000	230·2	− 5·1	2
VII	9 45	1538	241·0	+ 6·2	1
VIII	11 15	1179	254·9	+13·7	2
IX	12 45	1271	270·9	+16·2	2
X	14 15	1382	286·8	+13·1	1
XI	15 45	1773	300·4	+ 5·1	1
XII	17 15	1817	310·8	− 6·5	1
XIII	18 45	1172	318·1	−20·5	1
XIV	20 15	872	321·9	−35·8	1
XV	21 45	760	320·6	−51·4	2
XVI	23 15	746	307·6	−65·5	2

3. In each region the numbers of proper motions in sectors $\theta - 5°$ to $\theta + 5°$, where θ is the position-angle of a proper motion, were first obtained; this distribution was then smoothed by taking as the number of proper motions in any given 10° sector the mean of the counts in this and the two flanking sectors. Frequency-curves were then constructed and analysed by Eddington's "trial and error" method.

Denote by hV_1 and hV_2 the projections, on the tangent plane at the centre of a region, of the space-velocities hW_1 and hW_2 of the two drifts; θ_1 and θ_2 are the position-angles of the axes of the resolved drift-curves; N_1 and N_2 are the numbers of stars in a region belonging to Drift I and Drift II respectively.

The polar equation of a theoretical frequency-curve resulting from two drift-curves is

$$\rho = r_1 + r_2,$$

where

$$r_1 = 0{\cdot}04923 N_1 \phi(hV_1,\ \theta - \theta_1),$$
$$r_2 = 0{\cdot}04923 N_2 \phi(hV_2,\ \theta - \theta_2),$$

and ϕ is a function whose numerical values have been tabulated† by one of us. The procedure is the theoretically simple one of selecting suitable

* Several stars of the *Catalogue* have no measured proper motion; we have counted binaries as single stars, and to facilitate comparison with Jackson's discussion of his results which concern stars of known spectral type, we have omitted from our regions such stars for which the spectral type is not recorded (these stars amount to about 10 per cent. of the whole).

† W. M. Smart, *M.N.*, **87**, 128, 1926.

values of hV_1, hV_2, θ_1, θ_2, N_1 and N_2 such that the resulting frequency-curve will represent, as accurately as possible, the observed curve. In general we take $N_1 + N_2$ to be the total number of proper motions in the region concerned. The only exception is Region I.

The results of the analysis of the 16 regions are shown in Table II. In several of the regions the separation of the drifts as regards position-angle was achieved with a certain amount of uncertainty; in the later analysis such regions are given half-weight as compared with the others, as shown in Table I. As all the regions are well-populated it was considered inexpedient to assign weights according to the total number of stars per region; hence the weights to be used subsequently are those given in Table I.

The diagrams show the observed distributions of proper motions in position-angle with the theoretical distributions that appear to give the most satisfactory agreement.

TABLE II

Analysis of the Drift-Curves

Region	Drift I			Drift II		
	hV_1	θ_1	N_1	hV_2	θ_2	N_2
		°			°	
I	1·00	85	372	0·65	235	240
II	1·20	65	250	0·55	220	314
III	1·20	45	370	0·50	200	336
IV	1·10	10	345	0·40	190	474
V	0·95	335	680	0·45	190	432
VI	1·05	305	1360	0·45	190	640
VII	1·50	285	945	0·50	210	593
VIII	1·50	271¼	793	0·50	160	386
IX	1·70	255	850	0·55	160	421
X	1·60	237½	790	0·40	190	592
XI	1·45	220	1150	0·70	160	623
XII	1·70	195	1000	0·70	180	817
XIII	1·30	175	701	0·50	180	471
XIV	1·30	145	429	0·35	190	443
XV	1·30	117½	310	0·45	195	450
XVI	1·00	105	400	0·50	225	346

4. *The Drift-Constants.*—Let (α, δ) be the equatorial co-ordinates of the centre of a region. If (X, Y, Z) are the linear components of the space-velocity of a drift, relative to the Sun and with reference to the usual equatorial axes, we have for each region and for a particular drift the pair of equations of condition

$$X \sin \alpha - Y \cos \alpha = -hV \sin \theta,$$
$$X \cos \alpha \sin \delta + Y \sin \alpha \sin \delta - Z \cos \delta = -hV \cos \theta,$$

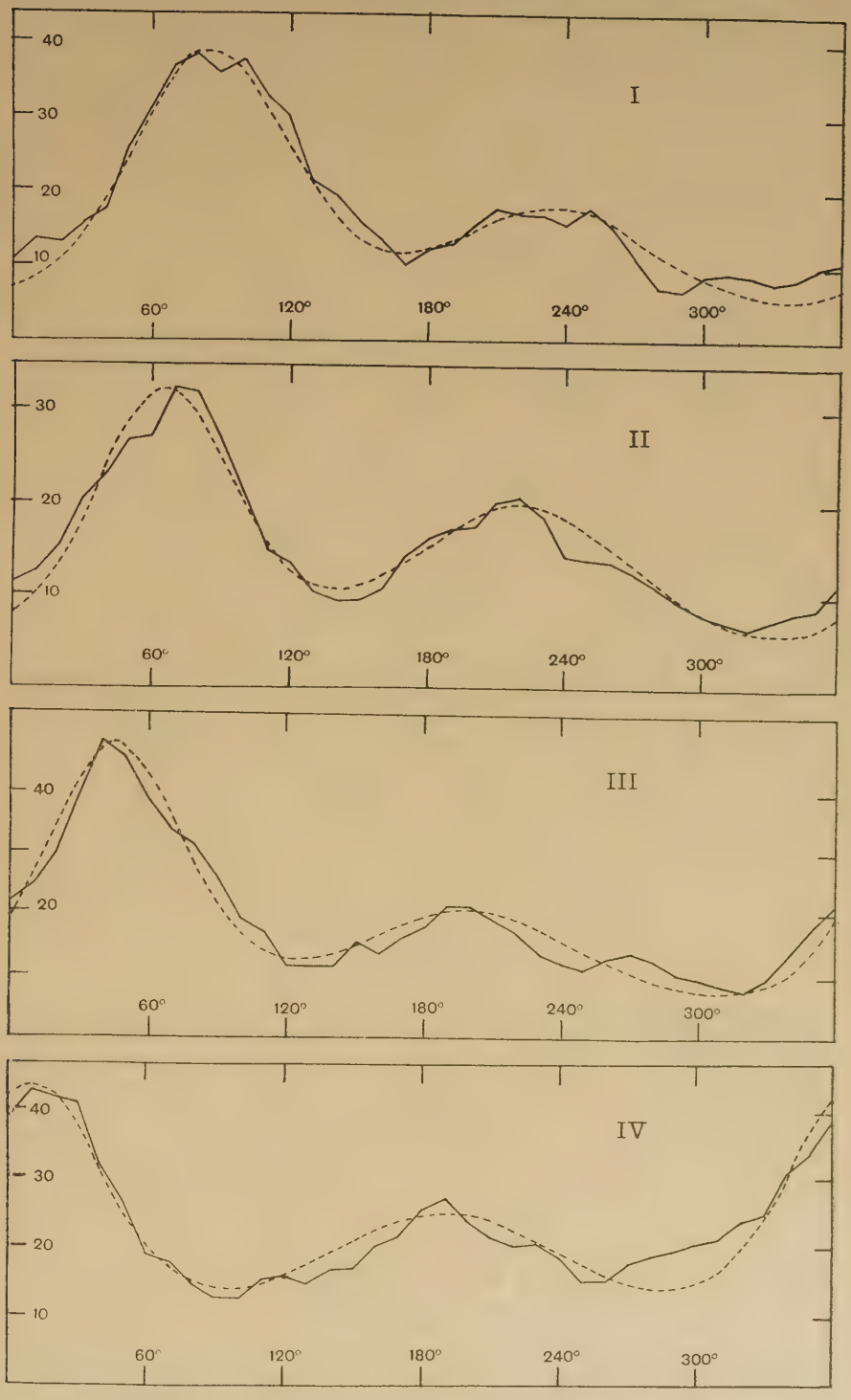

FIG. 1.—*Regions* I–IV. *Observed distribution—full line; theoretical distribution—broken line; abscissæ—position-angles; ordinates—numbers of stars.*

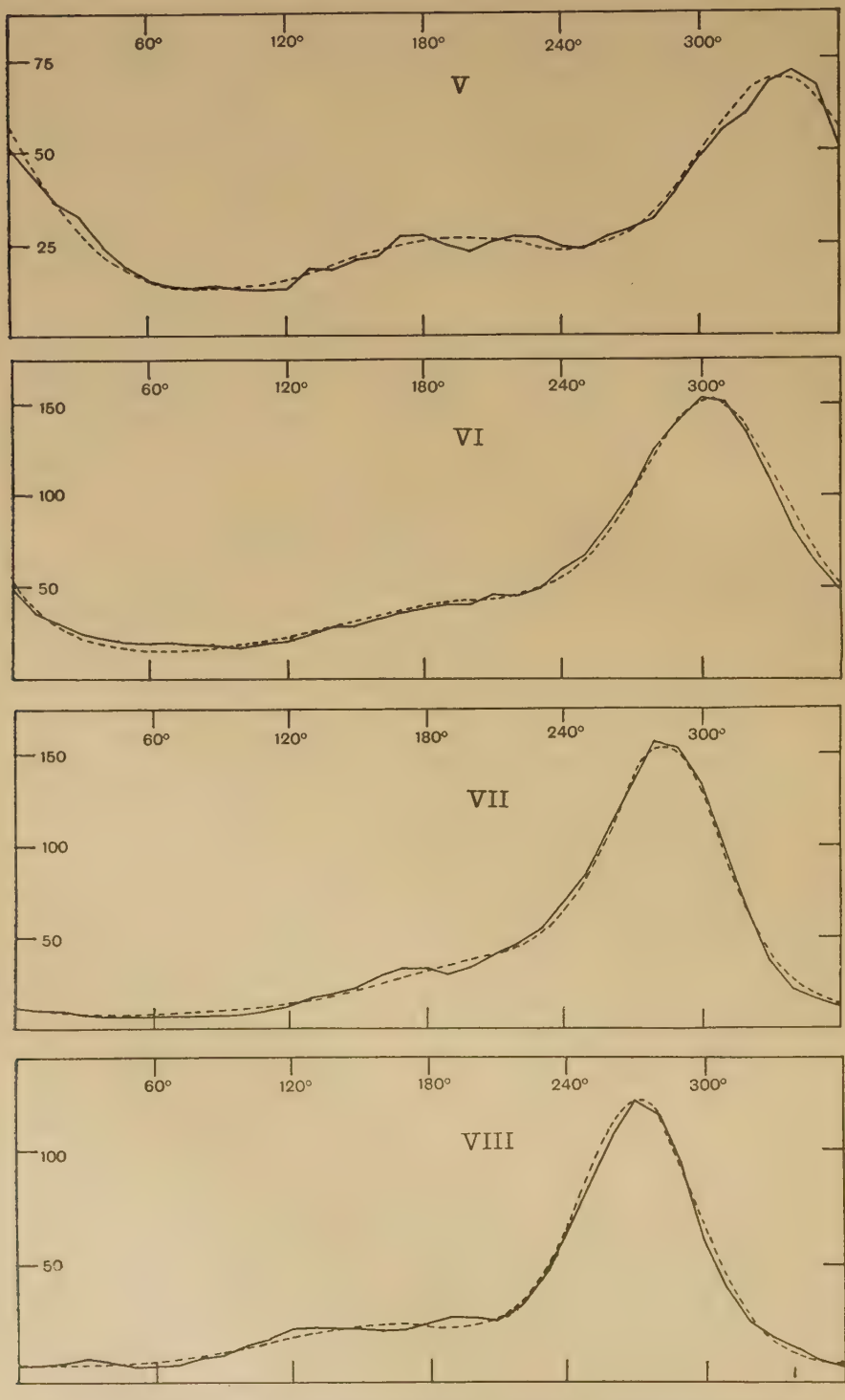

FIG. 2.—*Regions* V–VIII. *Observed distribution—full line; theoretical distribution—broken line; abscissæ—position-angles; ordinates—numbers of stars.*

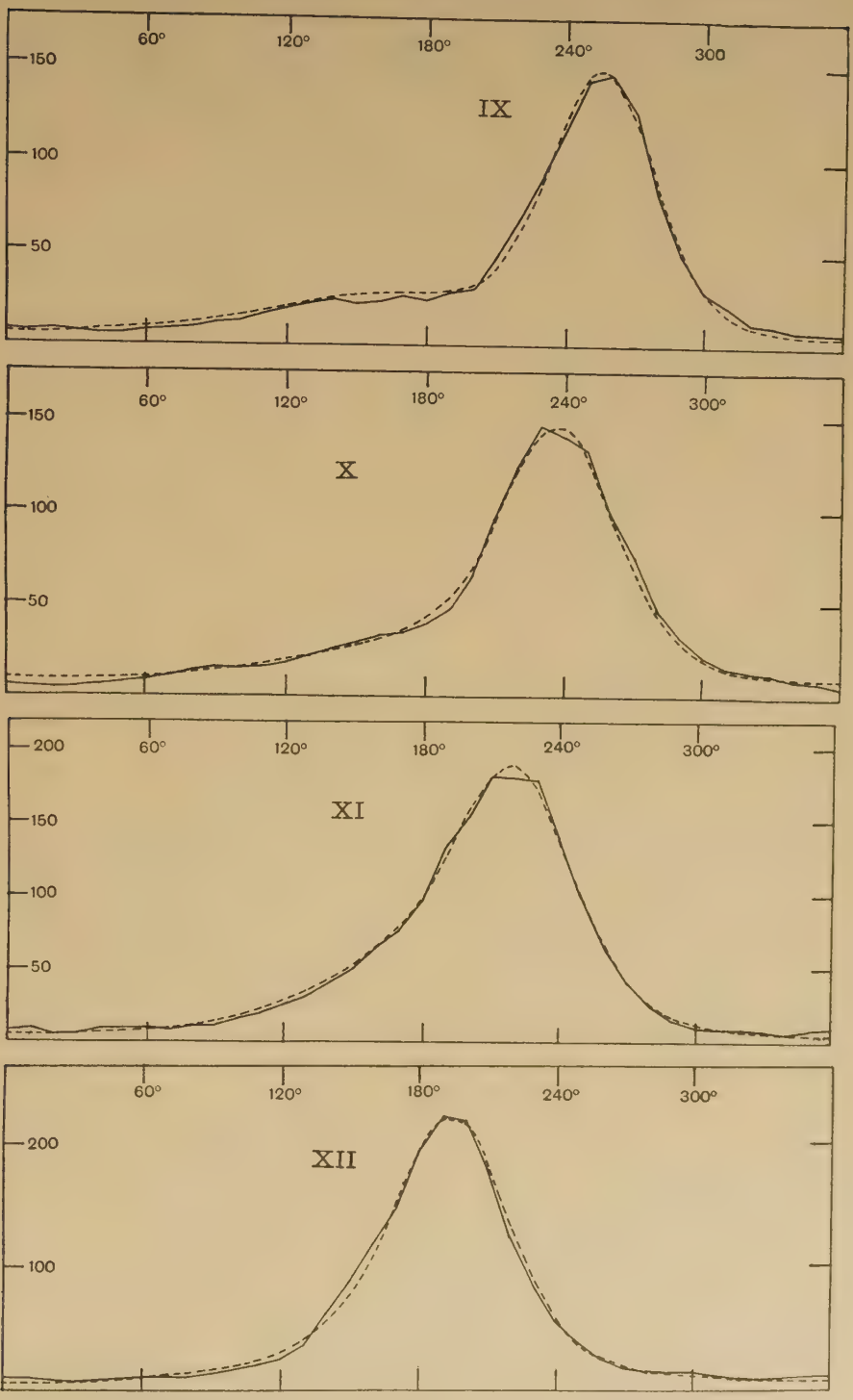

FIG. 3.—*Regions* IX–XII. *Observed distribution—full line; theoretical distribution—broken line; abscissæ—position-angles; ordinates—numbers of stars.*

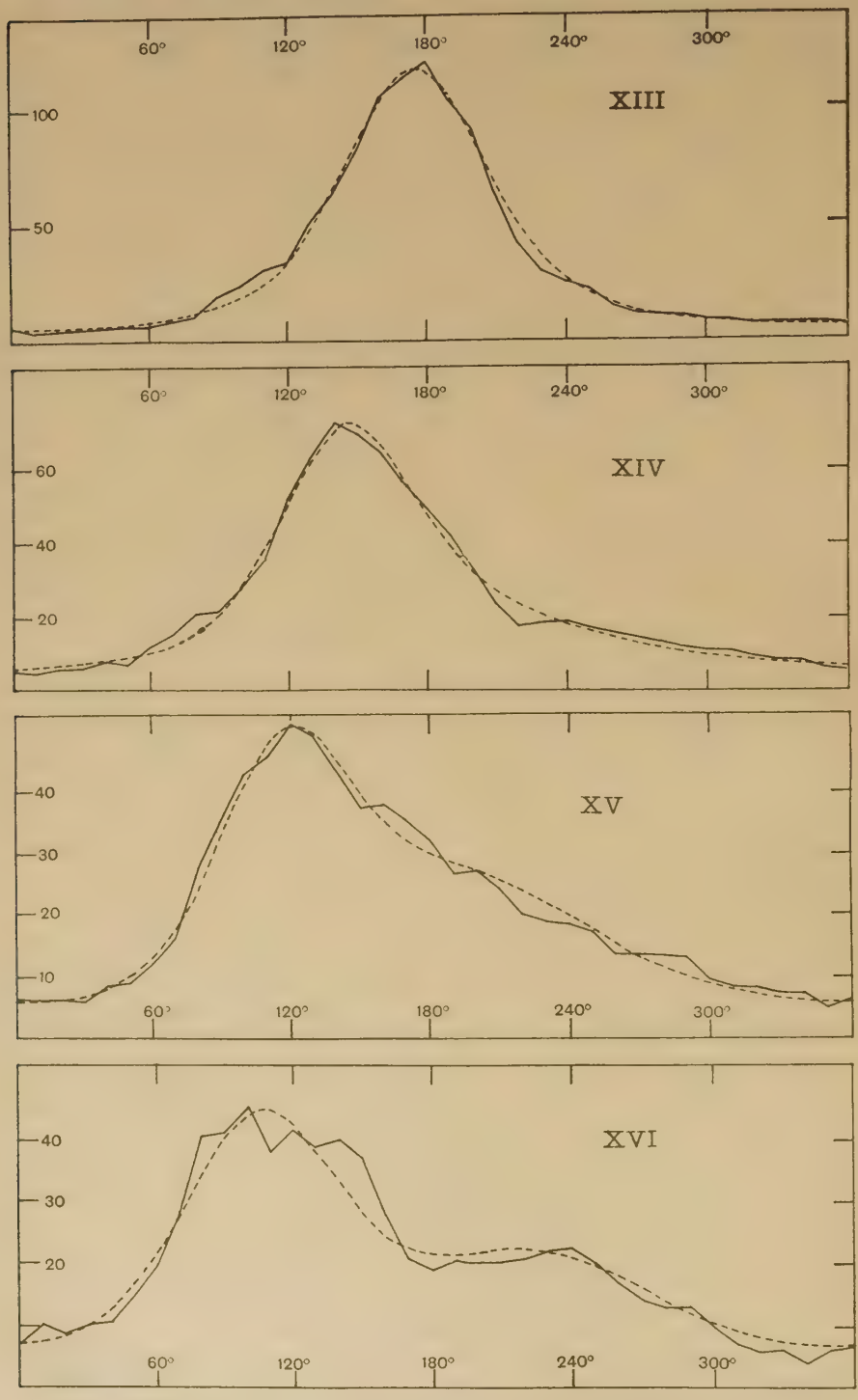

FIG. 4.—*Regions* XIII–XVI. *Observed distribution—full line; theoretical distribution—broken line; abscissæ—position-angles; ordinates—numbers of stars.*

in which hV and θ are respectively the projected drift-velocity and the position-angle of the axis of the drift, their values being given in Table II. Writing $Z_1 = Z \cos \delta$, the final normal equations obtained are :

		Drift I	Drift II
$9{\cdot}615X$	$-0{\cdot}596Z_1 =$	$+\ 0{\cdot}411$	$+0{\cdot}636$
$10{\cdot}06Y - 1{\cdot}299Z_1 =$		$+14{\cdot}342$	$-1{\cdot}978$
$-0{\cdot}596X - 1{\cdot}299Y + 13Z_1 =$		$+\ 0{\cdot}06$	$+5{\cdot}82$

from which we obtain the following results * :—

	Drift I	Drift II
X:	$+0{\cdot}052 \pm 0{\cdot}035$	$+0{\cdot}093 \pm 0{\cdot}029$
Y:	$+1{\cdot}446 \pm 0{\cdot}032$	$-0{\cdot}140 \pm 0{\cdot}026$
Z:	$-0{\cdot}217 \pm 0{\cdot}044$	$-0{\cdot}628 \pm 0{\cdot}036$

The equatorial co-ordinates (A, D) of the drift-apices are then found from the formulæ

$$\tan A = Y/X, \qquad \tan D = Z/(X^2 + Y^2)^{1/2},$$

and the space-velocities, hW, of the drifts from

$$hW = (X^2 + Y^2 + Z^2)^{1/2}.$$

The results are :

	Drift I	Drift II
R.A. of Apex:	$88^\circ{\cdot}0 \pm 1^\circ{\cdot}4$	$303^\circ{\cdot}6 \pm 9^\circ{\cdot}5$
Dec. of Apex:	$-8^\circ{\cdot}5 \pm 0^\circ{\cdot}6$	$-75^\circ{\cdot}0 \pm 13^\circ{\cdot}5$
hW:	$1{\cdot}463 \pm 0{\cdot}035$	$0{\cdot}650 \pm 0{\cdot}035$

As regards Drift I the probable errors are satisfactorily small, and the definitive results are in good accordance with the constants determined from previous examinations of a similar nature (see Table V). As regards Drift II the co-ordinates of the apex are poorly determined. This is due in part to the high southerly declination of the apex and in part to the fact that, with a high southerly declination, the zonal regions give a small range in the values of θ_2 and λ_2, a small range in λ_2 leading to a small range in the values of hV_2. If we refer to Table III, the calculated values of θ_2 range between 158° and 202° only, whereas the corresponding values of θ_1 cover the complete range of 360°. In the same way the inconsiderable range of λ_2 (or of hV_2) is shown in Table IV.

With the above values of the apical co-ordinates we calculate the values of the position-angles for the several regions and for the two drifts. These are shown in Table III in the second and sixth columns ; the third and seventh columns contain the observed values of the position-angles obtained from the analysis of the frequency-curves of the proper motions. The agreement for Drift I is very satisfactory, but the Drift II residuals show several large values, notably so for the neighbouring Regions I and XVI, where the position-angles of the drift-apices have apparently a considerable separation.

* In these and later calculations we retain an additional place of decimals.

We can illustrate the convergence of the great circles defined by the observed values of the position-angles of the drift-apices as follows. Let β be the angular distance of an apex from such a great circle. Then, if θ_c and θ_o denote the calculated and observed values of the position-angle concerned,

$$\sin \beta = \sin \lambda \sin (\theta_c - \theta_o).$$

The values of β are shown in the fifth and ninth columns of Table III. So far as Drift I is concerned, the great circles all pass within 10° of the apex of the drift and, with one exception, within 5°. As regards Drift II, the convergence is less striking, as ten only of the great circles pass within 10° of the corresponding drift-apex.

TABLE III

The Calculated and Observed Values of the Position-Angles

Region	Drift I				Drift II			
	Cal.	Obs.	Diff.	β_1	Cal.	Obs.	Diff.	β_2
	°	°	°	°	°	°	°	°
I	86·5	85	+ 1·5	1·4	201·7	235	−33·3	20·8
II	68·9	65	+ 3·9	3·4	201·1	220	−18·9	13·5
III	46·1	45	+ 1·1	0·8	197·8	200	− 2·2	1·7
IV	14·9	10	+ 4·9	3·0	192·8	190	+ 2·8	2·3
V	338·9	335	+ 3·9	2·4	186·5	190	− 3·5	3·0
VI	309·3	305	+ 4·3	3·2	180·0	190	−10·0	8·6
VII	287·8	285	+ 2·8	2·5	173·2	210	−36·8	30·6
VIII	270·6	271¼	− 0·6	0·6	167·1	160	+ 7·1	5·9
IX	254·6	255	− 0·4	0·4	162·1	160	+ 2·1	1·6
X	237·2	237½	− 0·3	0·3	158·9	190	−31·1	21·8
XI	216·4	220	− 3·6	3·1	158·4	160	− 1·6	1·0
XII	191·2	195	− 3·8	3·1	161·4	180	−18·6	10·5
XIII	164·1	175	−10·9	9·0	178·8	180	− 1·2	0·6
XIV	139·6	145	− 5·4	4·8	181·7	190	− 8·3	4·1
XV	119·5	117½	+ 2·0	1·9	191·4	195	− 3·6	1·8
XVI	102·5	105	− 2·5	2·5	198·8	225	−26·2	14·7

Table IV shows the comparison between the values of the projected drift-velocity calculated from the formula

$$hV = hW \sin \lambda$$

and the values obtained in the analysis of the frequency-curves. The values of λ_1 and λ_2 in the second and sixth columns are calculated from the positions of the drift-apices already obtained. As regards Drift I, Regions I and XVI again show large discrepancies.

TABLE IV

The Calculated and Observed Values of the Projected Drift-Velocities

Region	Drift I				Drift II			
	λ_1	hV_1			λ_2	hV_2		
		Cal.	Obs.	Diff.		Cal.	Obs.	Diff.
	°				°			
I	74·7	1·40	1·00	+0·40	40·4	0·42	0·65	−0·23
II	59·4	1·26	1·20	+0·06	46·2	0·47	0·55	−0·08
III	46·2	1·05	1·20	−0·15	51·5	0·51	0·50	+0·01
IV	38·1	0·90	1·10	−0·20	55·6	0·53	0·40	+0·13
V	39·0	0·92	0·95	−0·03	58·3	0·55	0·45	+0·10
VI	48·3	1·09	1·05	+0·04	59·2	0·56	0·45	+0·11
VII	62·0	1·29	1·50	−0·21	58·3	0·55	0·50	+0·05
VIII	77·5	1·42	1·50	−0·08	55·6	0·53	0·50	+0·03
IX	93·0	1·46	1·70	−0·24	51·4	0·51	0·55	−0·04
X	107·3	1·39	1·60	−0·21	46·1	0·47	0·40	+0·07
XI	118·7	1·28	1·45	−0·17	40·3	0·42	0·70	−0·28
XII	125·1	1·20	1·70	−0·50	34·8	0·37	0·70	−0·33
XIII	124·4	1·20	1·30	−0·10	30·7	0·33	0·50	−0·17
XIV	117·0	1·30	1·30	0·00	29·2	0·32	0·35	−0·03
XV	104·9	1·41	1·30	+0·11	30·8	0·33	0·45	−0·12
XVI	90·3	1·46	1·00	+0·46	35·0	0·37	0·50	−0·13

5. *The Vertex of Star-Streaming.*—If (ξ, η, ζ) denote the linear components of the space-velocity of Drift I relative to Drift II, we obtain

$$\xi \equiv X_1 - X_2 = -0{\cdot}041 \pm 0{\cdot}046,$$
$$\eta \equiv Y_1 - Y_2 = +1{\cdot}586 \pm 0{\cdot}041,$$
$$\zeta \equiv Z_1 - Z_2 = +0{\cdot}411 \pm 0{\cdot}057,$$

from which we find (taking that vertex in the neighbourhood of the direction of the galactic centre):

R.A. of Vertex: $271^\circ{\cdot}5 \pm 1^\circ{\cdot}6$,
Dec. of Vertex: $-14^\circ{\cdot}5 \pm 2^\circ{\cdot}0$.

In galactic co-ordinates this position is:

Longitude of Vertex: $343^\circ{\cdot}2 \pm 1^\circ{\cdot}0$,
Latitude of Vertex: $+0^\circ{\cdot}5 \pm 1^\circ{\cdot}7$.

The Cape proper motions thus give a vertex lying on the galactic equatorial plane but, in common with previous proper motion results, greater in longitude by 16° than the direction of the galactic centre derived by other methods. The longitude is almost exactly the same as Jackson's value (see Table V).

The space-velocity of one drift relative to the other is $(\xi^2+\eta^2+\zeta^2)^{1/2}$, the value of which is

$$1{\cdot}638 \pm 0{\cdot}043.$$

6. *The Solar Motion.*—According to the results of our analysis (Table II), the numbers of stars in Drifts I and II are respectively 10,745 and 7578, giving the ratio 1·41 : 1. This numerical value of the ratio is affected by the uncertainty in the determination of the constants for Drift II due to the circumstances already alluded to, and for this reason is probably less reliable than results obtained from the analysis of regions more suitably situated with respect to the apex of Drift II in particular. Where such conditions are more favourable several investigations of the fainter stars give an approximate equality in the populations of the two drifts ; it is to be remarked, however, that the ratio found by Eddington from his analysis of the *P.G.C.* stars is 1·5 : 1.

Let hU_0 denote the solar motion with respect to the totality of stars and (A_0, D_0) the co-ordinates of the solar apex. Writing p for 1·41 we have :

$$\begin{aligned}
(p+1)hU_0 \cos A_0 \cos D_0 + phW_1 \cos A_1 \cos D_1 + hW_2 \cos A_2 \cos D_2 &= 0,\\
(p+1)hU_0 \sin A_0 \cos D_0 + phW_1 \sin A_1 \cos D_1 + hW_2 \sin A_2 \cos D_2 &= 0,\\
(p+1)hU_0 \sin D_0 + phW_1 \sin D_1 + hW_2 \sin D_2 &= 0,
\end{aligned}$$

the numerical values of the quantities in the second and third terms of each equation being given in section 4.

The solution of these equations is :

R.A. of Solar Apex (A_0) :	265°·1,
Dec. of Solar Apex (D_0) :	+26°·1,
Solar Speed (hU_0) :	0·879.

The feature of these results is the value of the declination, which is considerably lower than most of the values previously found in investigations of faint stars by similar methods. It should, however, be remarked that owing to the high value of D_2 most of the uncertainty attached to the determination of the position of the Drift II apex is thrown into the declination of the solar apex. Jackson also obtained a low value for the declination of the solar apex (see Table V below), which he attributed * to the introduction of the effect of galactic rotation in deriving the corrections required to convert the relative proper motions into absolute proper motions.

7. *Comparison of Results.*—In this section we compare our values of the stream-constants with those of several investigations based on the same method, and also with Jackson's results, from the same material, obtained from the application of Schwarzschild's ellipsoidal hypothesis. In Table V the columns (*a*) to (*e*) refer as follows :—

(*a*) *P.G.C.* stars.†

(*b*) Cambridge photographic proper motions of 3029 stars.‡

* *Cape Catalogue*, Introduction, p. xxix.

† A. S. Eddington, *M.N.*, **71**, 4, 1910.

‡ W. M. Smart, *M.N.*, **87**, 122, 1926.

(*c*) Groningen photographic proper motions of 4245 stars.*
(*d*) Present investigation.
(*e*) Jackson's results (ellipsoidal hypothesis).

TABLE V

Comparison of Results

		(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
Drift I	R.A. of Apex	90°·8	88°·4	89°·1	88°·0	...
	Dec. „ „	− 14°·6	− 11°·8	− 3°·3	− 8°·5	...
	hW_1	1·516	1·497	1·249	1·463	...
Drift II	R.A. of Apex	287°·8	289°·0	279°·0	303°·6	...
	Dec. „ „	− 64°·1	− 73°·3	− 56°·5	− 75°·0	...
	hW_2	0·855	0·975	0·607	0·650	...
Vertex	R.A.	94°·2	91°·7	91°·2	91°·5	90°·2
	Dec.	+ 11°·9	+ 19°·9	+ 15°·4	+ 14°·5	+ 13°·4
	Gal. Long.	346°·8	338°·5	342°·4	343°·2	343°·6
	Gal. Lat.	− 0°·6	− 2°·4	+ 0°·3	+ 0°·3	+ 2°·1
	Rel. velocity of drifts	1·868	1·840	1·637	1·638	...
Solar Motion	R.A. of Apex	267°·3	273°·2	272°·6	265°·1	266°·9
	Dec. „ „	+ 36°·4	+ 43°·6	+ 31°·1	+ 26°·1	+ 28°·3
	hV_0	0·908	0·881	0·553	0·879	...
	N_1/N_2	1·48	1·10	1·04	1·41	...

As regards the vertex of star-streaming, the galactic co-ordinates refer to that direction in the neighbourhood of the galactic centre, while the equatorial co-ordinates refer to the opposite direction. The last line gives the ratio of the populations of the two drifts.

It is to be remarked that two-thirds of the photographic regions investigated in (*c*) are within a narrow zone of declination − 45°, which is practically the same as the declination of the centre of the zone in (*d*).

8. *Calculation of the Ellipsoidal Constants from the Stream-Constants.*— In a recent paper † it was shown how the ellipsoidal constants could be derived from the assumed stream-constants, and the results were then compared with the ellipsoidal constants obtained from an investigation of radial velocities. In the present paper we use our values of the stream-constants in a similar calculation, and compare our results with Jackson's values obtained from the same proper-motion material.

If ϵ denotes the ratio of the minor axis to the major axis of the velocity ellipse in a given region, we have, with the notation of the present paper,

$$q \equiv \frac{1}{\epsilon^2} - 1 = h^2V_1^2 + h^2V_2^2 - 2h^2V_0^2 \sin^2 \lambda,$$

* G. H. ten Bruggencate, *B.A.N.*, No. 85, 1925; W. M. Smart, *M.N.*, **88**, 144, 1927.

† W. M. Smart, *M.N.*, **99**, 561, 1939.

where λ denotes the angular distance of the region from the solar apex. This equation enables us to calculate the values of q for the several regions.

If K/H is the ratio of the minor axis to the major axis of the velocity ellipsoid and $x \equiv \frac{H^2}{K^2} - 1$, we have the equation of condition

$$q = x \sin^2 \chi,$$

where χ is the angular distance of a region from the vertex.

From the 16 equations of this form we require to find x and the longitude G_0 of the vertex (we assume that the vertex lies on the galactic equator). As explained in the paper referred to, the procedure is to assume approximate values x_0 and G_0' ; the equation of condition can then be written

$$a\Delta x + b\Delta G = q - q_0,$$

where

$$\Delta x = x - x_0, \qquad \Delta G = G_0 - G_0', \qquad q_0 = x_0 \sin^2 \chi_0,$$
$$a = \sin^2 \chi_0, \qquad b = x_0 \sin 2(G - G_0') \cos^2 g ;$$

in these χ_0 is the value of χ calculated for the approximate value, G_0', of G_0, and (G, g) are the galactic co-ordinates of the centre of a region (the last are given in Table I).

A first solution was made with $x_0 = 1{\cdot}05$ and $G_0' = 343°$, the results of which suggested a closer approximation as follows :—

$$x_0 = 1{\cdot}0, \qquad G_0' = 336°.$$

With these values a second solution was carried out. The numerical data are shown in Table VI ; with the value of x_0 adopted, the values of q_0 are the same as the values of a as tabulated.

The normal equations are

$$9{\cdot}600\Delta x + 0{\cdot}330\Delta G = \quad 0{\cdot}115,$$
$$0{\cdot}330\Delta x + 4{\cdot}518\Delta G = -0{\cdot}107,$$

from which

$$\Delta x = +0{\cdot}0128 \pm 0{\cdot}0142,$$
$$\Delta G = -0{\cdot}0246 \pm 0{\cdot}0207.$$

The probable errors are derived from the residuals shown in Table VI.

The definitive values of x and G_0 are then

$$x = 1{\cdot}013 \pm 0{\cdot}014,$$
$$G_0 = 334°{\cdot}6 \pm 1°{\cdot}2.$$

From the first of these we find

$$\frac{K}{H} = 0{\cdot}705 \pm 0{\cdot}002.$$

It is to be remarked that the small probable errors in G_0 and K/H merely indicate the high degree of accuracy of the relationship between the stream and ellipsoidal constants ; they do not, of course, include the uncertainty

TABLE VI

Region	a	b	q	$q - q_0$ (Unit 0·001)	Residual (Unit 0·001)
I	0·988	+0·063	1·028	+ 40	− 30
II	0·985	−0·101	1·110	+125	−101
III	0·913	−0·340	1·025	+112	− 93
IV	0·852	−0·566	0·915	+ 63	− 39
V	0·859	−0·653	0·866	+ 7	+ 19
VI	0·926	−0·520	0·888	− 38	+ 61
VII	0·993	−0·172	0·939	− 54	+ 70
VIII	0·977	+0·288	0·932	− 45	+ 41
IX	0·837	+0·704	0·862	+ 25	− 32
X	0·595	+0·938	0·626	+ 31	− 47
XI	0·344	+0·940	0·368	+ 24	− 43
XII	0·192	+0·761	0·179	− 13	− 3
XIII	0·205	+0·513	0·118	− 87	+ 77
XIV	0·381	+0·311	0·279	−102	+ 99
XV	0·638	+0·200	0·556	− 82	+ 84
XVI	0·868	+0·143	0·853	− 15	+ 22

arising from the definitive values of the stream-constants used in the calculations.

Jackson's value of G_0, as already stated in Table V, is 343°·6, and his value of K/H is 0·94/1·60 or 0·59; these values refer to all the stars in the zone.

Both values differ appreciably from the results obtained by the method of this section. The discrepancies, however, are comparable with those found by Jackson in treating stars in the two halves of the zone from 6^h to 18^h and from 18^h to 6^h, containing respectively two-thirds and one-third approximately of the total number of stars. The corresponding values of the galactic co-ordinates of the vertex and of K/H are as follows :—

6^h to 18^h : 348°·0, −1°·4; 0·73,
18^h to 6^h : 338°·5, −5°·9; 0·45.

A similar discrepancy is found in the co-ordinates of the solar apex which, as already mentioned, Jackson attributes to the application of the corrections for galactic rotation in a limited region of the sky.

A very close agreement between the values of the ellipsoidal constants derived by two such wholly dissimilar methods—the method of this section and that depending on the analysis of the proper motions according to Schwarzschild's ellipsoidal hypothesis—is, perhaps, hardly to be expected in the present instance. As we have seen, our determination of Drift II carries a large uncertainty, due primarily to the situation of the zone, which is reflected first in the co-ordinates of the solar apex and subsequently in the calculated values of q applied in the method of this section. Consequently, our deduced value of K/H, in particular, must carry a substantial probable error the value of which it is almost impracticable to estimate. It would appear from the data just given that a similar remark applies to Jackson's determination of K/H.

Summary

1. The Cape photographic proper motions of 18,323 stars in the zone $-40°$ to $-52°$ and grouped together in 16 regions are analysed according to the two-streams theory.

2. The constants of Drifts I and II are given in section 4.

3. The galactic co-ordinates of the vertex are ($343°{\cdot}2$, $+0°{\cdot}5$).

4. The equatorial co-ordinates of the solar apex are ($265°{\cdot}1$, $+26°{\cdot}1$).

5. The ellipsoidal constants are calculated from the derived values of the stream-constants.

6. The results are compared with Dr. Jackson's results obtained from the analysis of the proper motions according to the ellipsoidal hypothesis.

University Observatory,
University Gardens,
Glasgow, W.2 :
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THE SUN'S SPEED OF GALACTIC ROTATION DETERMINED FROM THE GLOBULAR CLUSTERS

G. L. Camm, Skynner Student in Astronomy, Balliol College, Oxford

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1. *Introduction.*—In any attempt to determine the mass of the galaxy by dynamical methods it is necessary to know the Sun's speed of galactic rotation, the value of which has generally been taken to be between 270 and 300 km./sec. The value has been determined from the motion of the Sun relative to the globular clusters, in the first place, by Strömberg *, and, more recently, by Edmondson.†

The method has assumed that the mean motion of the globular clusters is zero, so that the clusters form a non-rotating system. However, the analysis of Mineur ‡ has shown that this assumption is unlikely to be correct. Mineur's theory has been severely criticized by Edmondson §, but his belief that the clusters share in the general galactic rotation has never been disproved.

The analysis which follows reveals that the radial velocities of the clusters are by no means inconsistent with a differential rotation of the whole system. If this effect is accepted, then the value of the Sun's speed of galactic rotation comes out as 410 km./sec. ; this value may only be a lower limit.

2. *The Differential Rotation of the Clusters.*—The radial velocities of 26 globular clusters are given by Edmondson ||, who has used them to find the velocity of the Sun relative to the group as a whole. In place of his assumption that the mean motion of the clusters is zero, it was supposed that there is a mean rotation about the galactic axis. Such rotation, if it exists, may vary with R, the distance from the galactic axis, and may thus produce differential effects. There will probably also be some slight variation with the distance from the galactic plane, but in view of the very limited number of observations, this effect has had to be overlooked.

The galactic centre was taken to be at a distance of 10,000 parsecs from the Sun in the direction of galactic longitude, 333°. It was convenient to adopt a frame of co-ordinates which rotated about the galactic axis ; the speed of rotation was considered to be chosen so that the motion of the Sun in the moving frame had only the two components, u towards the galactic centre, and w towards the galactic pole. In this moving frame the mean motion of the clusters at a distance R from the axis is rotation

* *Mt. Wilson Contr.* No. 292, 199, 1925.
† *A.J.*, 45, 2, 1935.
‡ *M.N.*, 96, 61, 1935.
§ *M.N.*, 96, 636, 1936.
|| *A.J.*, 45, 1, 1935.

about that axis with angular velocity, $\omega(R)$, in the direction of decreasing galactic longitude. On these assumptions we should expect that a cluster which is at a distance R from the axis, and which is seen from the Sun in the direction of galactic co-ordinates (l, b) would have for its radial velocity a quantity

$$R_0\omega(R) \sin (l-l_0) \cos b - u \cos (l-l_0) \cos b - w \sin b.$$

In this, R_0 and l_0 are the distance from the Sun and the longitude of the galactic centre. It follows that, if ρ is the observed radial velocity of such a cluster, the quantity $\omega(R)$ can be determined from equations of the type

$$\rho + u \cos (l-l_0) \cos b + w \sin b - R_0\omega(R) \sin (l-l_0) \cos b = 0. \qquad (1)$$

If, as Edmondson has supposed, the mean motion of the clusters is zero at all points, then ω is independent of R. In this case the solution would be carried out exactly as Edmondson has done, and the values obtained—Edmondson's values, suitably projected—would be

$$u = -18{\cdot}4 \text{ km./sec.}, \qquad w = +3{\cdot}6 \text{ km./sec.}, \qquad R_0\omega = -274{\cdot}1 \text{ km./sec.}$$

However, the possibility of differential rotation cannot be overlooked, and for this reason it is necessary to examine equation (1) in the case when ω varies with R.

The distances of the objects from the Sun were originally given by Shapley *, and have been corrected by van de Kamp † for interstellar absorption in a layer of total thickness, 0·8 mag. This absorption is rather greater than the value of 0·5 mag. given by Hubble ‡ ; it is this latter value which has here been adopted.

As there are only 26 objects available, as the values of R extend up to 38,000 parsecs, and as the observed radial velocities are considered to be uncertain to 50 km./sec., it was considered quite impracticable to determine the function $\omega(R)$ by the method which has been used elsewhere for the planetary nebulæ. § The best that could be done was to divide the objects into three groups according to their distances, R, and then find the mean values of $R_0\omega(R)$ for each group. There were found to be six clusters between 38,000 and 12,000 parsecs of the galactic axis, ten between 10,000 and 6000 parsecs, and ten at distances less than 4000 parsecs. This sub-division depends very little on the amount of interstellar absorption assumed; if we had adopted van de Kamp's value, the absolute distances would all be reduced, but the objects would remain in the same sub-groups.

The solutions of equation (1) for the components of solar motion, u and w, and the three values of $R_0\omega(R)$, and σ, the root-mean-square residual velocity, are given in Table I. It will be seen that the values of u and w are in satisfactory agreement with those of Edmondson ; on the other hand, the quantity $R_0\omega(R)$—which must be independent of distance if

* *Star Clusters*, pp. 224–227, 1930.
† *A.J.*, **42**, 161, 1933.
‡ *Mt. Wilson Contr.* No. 485, 182, 1934.
§ *M.N.*, **99**, 82, 1938.

TABLE I

The Components of Solar Motion and the Mean Speeds of Rotation for the Subgroups

	Group I	Group II	Group III
Range of R (parsecs)	38,000–12,000	10,000–6000	4000–0
Mean value of R (parsecs)	24,500	8400	2900
No. of objects	6	10	10
u km./sec.	− 15·4 ± 22·3	− 15·4 ± 22·3	−15·4 ± 22·3
w km./sec.	− 15·6 ± 29·3	− 15·6 ± 29·3	−15·6 ± 29·3
$R_0\omega(R)$ km./sec.	−410·0 ± 49·7	−184·5 ± 48·3	+66·4 ± 138·8
σ km./sec.	75·5	125·1	104·3

Edmondson's determination of the Sun's rotational speed is to be valid—is found to decrease rather rapidly with distance. This seems to establish the existence of differential rotation for the clusters, of the same nature as is found in the motions of other bodies; indeed, there is even quantitative agreement, for we can roughly estimate Oort's constant, A, from this data, obtaining the value 17 km./sec. per 1000 parsecs as compared with the value 15·5 km./sec. per 1000 parsecs obtained from the B-type stars by Plaskett and Pearce.* The existence of differential rotation will invalidate Edmondson's determination of the Sun's speed of rotation in the galaxy, but another estimate now becomes possible.

3. *The Sun's Speed of Rotation.*—The method of finding a new value of the Sun's speed is quite simply to find the greatest difference between the angular speeds of rotation of the Sun and of other objects in the galaxy—preferably objects at great distances from the galactic axis, so that their own angular speeds are presumably small. This will determine at least a lower limit to the angular speed of the Sun, and hence a lower limit to the linear speed of the Sun. The globular clusters offer themselves as the most obvious choice, particularly on account of the very large values of $R_0\omega(R)$, which is a measure of the difference in angular speeds which we wish to determine.

The mean values of $R_0\omega(R)$ for the three groups have already been determined; the next step was to suppose that $R_0\omega(R)$ varied linearly with R within each group, and to determine the constants of proportionality. For the two inner groups the range of R is no more than 4000 parsecs, so that, on account of the large velocity-dispersion, the rate of variation of ω with R was indeterminate for these two groups. On the other hand, the six clusters in the outermost group are scattered over a range of 26,000 parsecs and have a smaller velocity-dispersion, so that here a determination was possible. The value of $R_0\frac{d\omega}{dR}$ came out as $-0{\cdot}18 \pm 5{\cdot}09$ km./sec. per 1000 parsecs. As the mean value of $R_0\omega(R)$ for this group is -410 ± 50 km./sec., the variation of ω within the group is quite negligible, and $R_0\omega$ is effectively constant over the entire range of 26,000 parsecs. It

* *Publ. Dom. Ap. Obs.*, 5, 300, 1933.

should be pointed out that this is the only stage at which the position of the galactic centre and the actual distances of the objects were really required. The value of $-0{\cdot}18$ is obtained when the thickness of the absorbing layer is taken to be 0·5 mag.; if the higher value of the absorption is adopted, the value of $R_0\frac{d\omega}{dR}$ is only decreased to about $-0{\cdot}22$ km./sec. per 1000 parsecs, so that our conclusion remains unaffected.

It has therefore been found that the mean motion of the clusters from 12,000 to 38,000 parsecs appears to be rotation as a rigid body in this moving frame of axes. On dynamical grounds it is more reasonable to believe that these six clusters form a non-rotating frame, and that their apparent motion is caused by the opposite rotation of the Sun around the galactic centre. As we have found an apparent angular velocity of the outer clusters of -410 ± 50 km./sec. per 10,000 parsecs, it appears that the speed of rotation of the Sun is $+410 \pm 50$ km./sec.

If the assumption that the outer clusters are not rotating is considered untenable, then it must be concluded that the speed of rotation exceeds 410 km./sec. The reason for this is that, on dynamical grounds, the mean rotation of objects at different distances will always have the same sense. But the angular velocity of the Sun exceeds that of the clusters by 410 km./sec. per 10,000 parsecs, so the absolute linear speed of rotation of the Sun must exceed 410 km./sec.

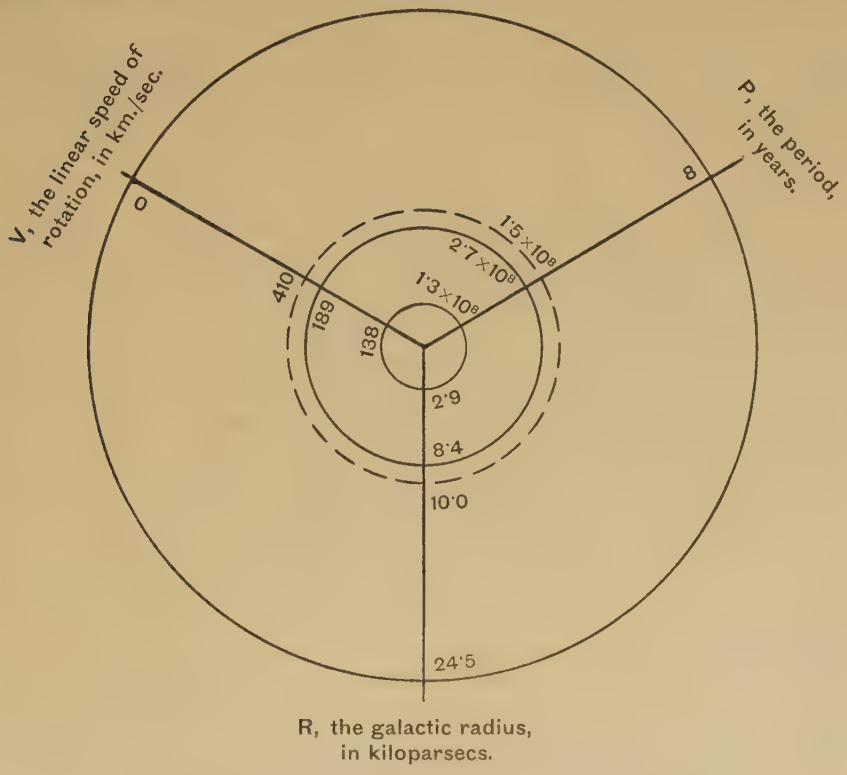
This high value of the Sun's speed is partly confirmed by Strömberg's determination * of the solar motion relative to the extra-galactic nebulæ. This determination is made difficult by the very large values of the K-term, but, of the seven values of the solar motion which Strömberg obtained by different groupings of the objects, only one is less than 360 km./sec., and three exceed 500 km./sec.

On the assumption that the outermost group of clusters is at rest, it is possible to deduce the mean angular and mean linear speeds of rotation of the inner groups. This is illustrated in the diagram, where the radii of the three full circles correspond to the mean distances of the groups from the galactic axis; the broken circle corresponds to the Sun's distance. Associated with each circle is the mean linear speed of rotation, V, and a period of rotation, P, which corresponds to the mean angular speed. It will be seen that the linear speed and period for the Sun differ greatly from the values of V and P which we should deduce for the clusters at 10,000 parsecs. This is consistent with Lindblad's theory †, in which he suggested that the galactic system was composed of sub-systems with different degrees of flattening and different characteristics in galactic rotation. The clusters were supposed to be the least flattened and to show the slowest speed of rotation; it was also suggested that the dispersion of the internal velocities of the sub-systems should increase from the most flattened to the least flattened system. All this, it will be seen, is fully confirmed so far as the globular clusters are concerned.

* *Mt. Wilson Contr.* No. 292, 199, 1925.

† *Arkiv för Matematik*, **19, A**, No. 21, 1925.

It must be emphasized that the rotation which we claim to have found does not imply that the motion of individual objects is a rotation about the galactic axis; this is only true of the mean motion. In the direction perpendicular to the galactic plane, and in the direction parallel to the plane and outwards from the axis, there is no reason to suppose that an individual object is more likely to be approaching than receding from the galactic centre, and so the mean motions in these directions are zero. The



The linear speeds and periods of rotation for the sub-groups of globular clusters.

large velocity-dispersion—it is 107 km./sec. for the 26 objects, as compared with Strömberg's value * of 113 km./sec. for 18 of them—will permit individual clusters to have high velocities in these directions, and will also allow a wide range of values of the linear speed of rotation, so that some of the objects may even rotate in the retrograde direction, as Edmondson has found.

In conclusion, it has to be admitted that the estimate of the Sun's speed of rotation at 410 ± 50 km./sec. has been made from most unsatisfactory material; in effect, only six objects have been used; their radial velocities are considered to be uncertain to 50 km./sec., and there is a velocity-dispersion of 75 km./sec. But it must be remembered that the previous determinations of Strömberg and Edmondson, though they employ rather more material,

* *Mt. Wilson Contr.* No. 292, 199, 1925.

have also used the uncertain radial velocities and show a much larger velocity-dispersion.

However, this paper will have served its purpose if it has shown that the existence of differential rotation among the clusters would make the earlier estimates of the Sun's speed rather unsatisfactory, and that the method which has here been carried out could be adopted as a safer alternative; and if the re-opening of the whole question of determining the Sun's speed from the globular clusters leads to the accurate measurement of radial velocities for many more of the clusters. Not until two or three times as many radial velocities are available can very much confidence be placed in any such determination of the Sun's speed of rotation.

Summary

The radial velocities of the globular clusters have been used by Edmondson to determine the solar motion relative to the clusters; this value of 274 km./sec. has generally been accepted as the linear speed of galactic rotation of the Sun. This conclusion will only be true if the clusters themselves are free from differential galactic rotation. With this in view, the clusters have been grouped according to their distances from the galactic axis, and the angular speed of rotation of each group relative to the Sun has been found from the radial velocities. The variation of the angular velocity among the groups indicates directly that the clusters themselves are also sharing in the differential rotation. If it is assumed that the outermost six clusters—which appear to be rotating as a rigid body—have no mean rotation, the linear speed of the Sun is found to be 410 ± 50 km./sec.; if the outer clusters are also rotating, then presumably the Sun's speed exceeds 410 km./sec.

The clusters are found to have a differential rotation, just as other types of object, but the speed of rotation is very much slower. This, together with the large velocity-dispersion and the spherical distribution of the objects in space, is all confirmation of Lindblad's theory of sub-systems within the galaxy, each with its own speed of rotation.

Again I am greatly indebted to Professor H. H. Plaskett and to Professor E. A. Milne for their continued interest and encouragement.

Oxford:
1939 *September.*

STELLAR PARALLAXES DETERMINED AT THE ROYAL OBSERVATORY, GREENWICH

(Communicated by the Astronomer Royal)

The following list is a continuation of the relative parallaxes obtained with the 26-inch refractor, and published annually in the November number of the *Monthly Notices*.

The stars are identified either by their numbers in the *Bonner Durchmusterung* or by references to other catalogues abbreviated as follows :—

A.C. = Greenwich Astrographic Catalogue.
Br. = Bradley.
C. = Carrington.
Gr. = Groombridge.
L. = Lalande.
L.F. = Lalande-Fedorenko.
$L.F._2$ = Lalande-Fedorenko supplement.
O.A. = Oeltzen-Argelander.
P. = Piazzi.
R. = Radcliffe, 1845·0.

The visual magnitude and spectral type for stars contained in the *Draper Catalogue* have been extracted from that catalogue. The photographic magnitudes are from the *Greenwich Astrographic Catalogue*.

Two values of the proper motion in R.A. are given. The first (as also the proper motion in Dec.) is absolute, and is based on meridian observations or the comparison of long-interval plates. The second is the relative proper motion derived from the parallax plates.

In the last column the first value of the probable error refers to the parallax, and the second to the relative proper motion in R.A. These are based on the value of the probable error for a plate of unit weight given in the eighth column. The mean probable error for unit weight is $\pm$ 0″·027.

To minimize the effect of quasi-seasonal error the observations have been spread over not less than six epochs. The average number of plates per star is 31 ; for those stars observed during the summer months the number is in excess of the average to compensate for the loss of factor.

Generally six comparison stars of mean magnitude 11–12 have been used, the magnitude of the parallax star being reduced to equality by means of a rotating sector. For the brighter stars a neutral-tinted filter absorbing 4·8 magnitudes is used in conjunction with the sector. Two exposures

of three minutes each are made on each plate, the carrier being turned through 180° after the first exposure. As the filter is permanently mounted on one of the carriers the reversal should eliminate any distortion due to the filter as well as to film displacement.

Star No. 7 is the fainter member of an optical pair, the other component (79° 169) being 4^m brighter and at a distance of only 15″. From a large number of plates, only those taken on nights of good definition were measured, all those which did not show a clear separation between the images being rejected. Even so, the large value for the probable error of unit weight reflects the difficulty of the measurements.

Attention should perhaps be drawn to star No. 27, whose parallax is +0″·198, although the total proper motion is only 92″ per century. Star No. 31 has a parallax of +0″·114 with a proper motion of 44″ per century. These faint stars were observed for parallax on account of the size of their proper motions. Nothing is known of their spectral types or colour indices, but they are presumably late-type dwarfs.

Star No. 28 is a close binary (see Aitken, *L.O.B.*, 413, 1929) for which a dynamical parallax of +0″·020 has been deduced.

Star No. 38 was found to have a large proper motion from some Edinburgh plates (Annual Report of the Astronomer Royal for Scotland, 1938). Examination of the Greenwich astrographic plates confirmed this discovery, and the values given in columns 6 and 7 were derived from measurement of these plates. The parallax given is preliminary; it is from two epochs only, assuming this value for the proper motion in R.A. observations of this star will be continued in order to derive a relative proper motion and to give a definitive parallax.

Star No. 39 is identical with No. 16 in the previous list (*M.N.*, **99**, 51, 1938). Two further epochs have been added in order to strengthen the determination, but although the probable errors have been reduced no significant alteration to the parallax is involved.

Star No. 41 is a faint binary whose components are of nearly the same magnitude and are separated by only 1″·5. The images were measured separately, but the measures were combined to give a mean result.

Attention is drawn to the erratum on page 277 of *M.N.*, **99**, 1939.

No.	Name and B.D. No.	Mag. Vis. Phot.	Spec. Type	R.A. Dec. 1900·0	Unit 0″·001 Proper Motion R.A.	Dec.	P.E. Unit Weight	Parallax	P.E.
				h m s					
1	Gr. 6	7·1	Ko	0 05 55	+ 72	− 16	24	+ 5	± 7
	73° 4	8·0		+73° 56′	+ 51				± 4
2	C. 145	6·4	Ko	0 59 06	+ 72	− 25	23	+ 7	± 6
	86° 17	7·0		+86° 37′	+ 42				± 3
3	L. 2981	8·0	F5	1 34 32	+191	− 9	27	+ 3	± 7
	68° 122	7·7		+68° 35′	+170				± 3
4	K. 519	8·4	Ko	3 26 49	− 12	− 2	23	+ 7	± 5
	79° 107	9·4		+79° 12′	− 23				± 3
5	L.F. 622	6·9	Ko	3 55 19	+ 28	− 43	20	−14	± 6
	74° 186	7·6		+74° 22′	+ 51				± 5
6	O.A. 5029	8·4	Ko	4 36 24	+ 17	− 51	23	+ 6	± 5
	69° 272	8·8		+70° 04′	+ 11				± 4
7	79° 168	9·3		5 06 06	+ 59	−162	38	+ 8	± 10
		9·5		+79° 07′	+ 53				± 5
8	A.C. 69° 2424	...		5 16 12	+ 54	−212	32	+19	± 8
		11·7		+69° 47′	+ 67				± 6
9	A.C. 74° 2116	...		5 27 48	+128	−233	29	+ 7	± 7
		11·5		+74° 06′	+111				± 6
10	A.C. 66° 2053	...		6 20 55	− 46	−389	21	−11	± 5
		11·5		+66° 29′	− 21				± 5
11	L.F. 1047–8	8·0	Ko	7 09 48	+ 44	− 37	22	− 3	± 5
	64° 628	8·2		+64° 18′	+ 44				± 5
12	L.F. 1027–8	7·3	Ko	7 10 08	+ 3	− 93	25	− 8	± 6
	78° 250	8·1		+78° 27′	+ 14				± 3
13	$L.F._2$ 57	7·8	Ko	7 13 44	− 9	0	24	+19	± 7
	84° 152	8·8		+84° 24′	+ 4				± 7
14	75° 314	8·9	Ko	7 37 07	− 15	− 50	23	− 2	± 6
		9·7		+75° 18′	+ 2				± 5
15	O.A. 8359	9·2	Ko	7 47 26	+ 48	− 28	14	− 1	± 3
	67° 512	9·6		+67° 42′	+ 37				± 3
16	78° 275	8·6	Ko	7 51 41	− 20	− 78	22	− 2	± 5
		8·8		+78° 33′	− 27				± 4
17	Gr. 1417	6·5	Ko	8 10 09	+ 51	+ 50	25	0	± 7
	75° 334	7·0		+75° 08′	+ 41				± 5

No.	Name and B.D. No.	Mag. Vis. Phot.	Spec. Type	R.A. Dec. 1900·0	Unit 0″·001 Proper Motion R.A.	Dec.	P.E. Unit Weight	Parallax	P.E.
				h m s					
18	L.F.$_2$ 83	7·4	Ko	8 23 44	+ 51	+ 33	21	+ 3	± 5
	73° 420	8·2		+73° 49′	+ 73				± 3
19	P.VIII 102	8·0	Ko	8 32 49	+ 10	− 80	23	+ 4	± 5
	65° 648	8·4		+65° 04′	− 6				± 3
20	L.F. 1378	7·2	Ko	8 41 04	− 49	+ 8	25	−21	± 6
	64° 707	8·3		+64° 38′	+ 7				± 6
21	A.C. 78° 3159	...		8 49 07	−185	−580	29	+ 2	± 7
		12·7		+78° 54′	−166				± 6
22	Gr. 1517	5·9	A2	9 05 50	− 77	− 60	25	+19	± 6
	73° 452	5·9		+73° 22′	− 74				± 2
23	A.C. 65° 3145	...		9 22 34	− 98	−207	20	+22	± 6
		11·8		+65° 59′	−119				± 6
24	Br. 1429	4·9	Ao	10 16 56	− 11	− 23	36	+38	± 9
	66° 664	5·0		+66° 04′	+ 31				± 4
25	A.C. 73° 4240	...		10 37 03	−357	+ 87	24	+27	± 7
		11·9		+73° 04′	−348				± 4
26	L.F. 1779	8·5	Ko	10 52 17	+ 1	− 92	25	−11	± 7
	65° 811	9·2		+65° 35′	− 2				± 4
27	A.C. 79° 3888	...		11 41 20	+730	+475	31	+198	± 9
		12·3		+79° 14′	+796				± 5
28	L.F. 1961	7·5	F5	11 48 43	− 59	− 17	24	+ 19	± 6
	72° 550	7·7		+72° 29′	− 68				± 3
29	A.C. 75° 4600	...		11 59 53	−278	− 75	25	+ 27	± 8
		11·3		+75° 54′	−265				± 4
30	O.A. 12784	8·8	Ko	12 30 58	− 50	+ 28	23	0	± 6
	76° 457	8·8		+76° 34′	− 46				± 4
31	A.C. 66° 3955	...		12 45 07	−433	− 86	26	+114	± 7
		12·2		+66° 40′	−434				± 5
32	R. 2946	7·2	Ko	12 53 09	+ 37	+ 1	26	+ 5	± 7
	83° 369	8·0		+83° 04′	+ 52				± 4
33	L.F. 2211	8·2	Ko	13 03 34	− 79	+ 27	23	+ 9	± 7
	78° 447	9·0		+78° 14′	− 79				± 4
34	A.C. 74° 4690	...		13 19 37	−438	+ 26	30	+ 13	± 8
		11·2		+74° 43′	−457				± 5

No.	Name and B.D. No.	Mag. Vis. Phot.	Spec. Type	R.A. Dec. 1900·0	Unit 0″·001 Proper Motion R.A.	Dec.	P.E. Unit Weight	Parallax	P.E.
				h m s					
35	76° 489	9·0	Ko	13 28 10	−118	− 10	26	+ 20	± 8
		9·2		+76° 08′	− 96				± 6
36	Gr. 2053	6·1	Ko	13 42 13	− 57	+ 62	26	+ 30	± 8
	78° 466	6·9		+78° 34′	− 59				± 4
37	68° 813	10·0		14 57 50	−139	+193	18	+ 1	± 5
		9·7		+68° 07′	−130				± 2
38	A.C. 76° 5308	...		15 43 15	+770	−780	27	+ 62	± 16
		13·0		+76° 20′	...				...
39	L.F. 2757	7·6	Ko	16 07 29	− 4	+ 56	24	+ 43	± 6
	81° 541	8·3		+80° 54′	− 15				± 2
40	26 Dra	5·3	F8	17 33 57	+242	−508	21	+ 52	± 6
	61° 1678	5·6		+61° 57′	+189				± 2
41	A.C. 73° 7443–4	...		19 32 47	−101 / − 85	− 74 / − 74	26	− 19	± 7
		12·3 / 12·1		+73° 15′	− 97				± 2
42	O.A. 26190	8·2	Ko	23 51 05	+ 27	+ 46	18	+ 12	± 5
	64° 1877	8·6		+64° 26′	+ 22				± 3

OCCULTATIONS OF STARS BY THE MOON OBSERVED AT MELBOURNE OBSERVATORY IN 1938

J. M. Baldwin

(Communicated by D. H. Sadler)

The observations of the occultations in this report were made by the eye-and-ear method throughout, the clock beating as a relay in the observing room, and the clock times were taken to $0^s.1$. The telescope used was an 8-inch refractor, the position being Long. $9^h\ 39^m\ 53^s.48$ E., Lat. $37°\ 49'\ 55''.5$ S., height above sea-level 31 metres. The values of $\sin\phi'$ and $\cos\phi'$, used in the reductions, were -0.61000 and 0.79082 respectively.

The reductions were performed in duplicate, using the method and reduction elements given in the *Nautical Almanac* for 1938. Under the heading *Serial No.* are the numbers given in the lists of Elements of Occultations and Reduction Elements in the *Nautical Almanac*.

The correction applied throughout to the observed time, in order to correct for the error in the Moon's tabular longitude, was $+0^h.00152$.

TABLE I

Serial No.	Star	Mag.	Phase	Date	U.T. h	m	s
135	B.D. +20° 1549	7·0	D	Feb. 11	14	43	46·8
195	B.D. +21° 1203	7·1	D	Mar. 10	12	11	13·5
214	B.D. +16° 1679	6·6	D	12	11	12	54·8
304	B.D. +8° 2285	7·1	D	Apr. 10	12	08	51·7
346	ρ Sagittarii	4·0	D	21	15	26	28·6
346	ρ Sagittarii	4·0	R	21	16	23	55·4
400	B.D. −0° 2442	6·3	D	May 9	15	50	28·1
406	B.D. −4° 3235	6·3	D	10	13	13	23·4
481	ψ Virginis	4·9	D	June 7	15	27	02·3
506	β^1 Scorpii	2·9	D	11	7	56	17·9
506	β^1 Scorpii	2·9	R	11	9	02	42·3
507	B.D. −19° 4308	5·1	D	11	7	56	23·0
562	B.D. −17° 4243	7·0	D	July 7	9	08	43·5
564	B.D. −17° 4273	7·0	D	7	15	25	59·2
750	B.D. −20° 4661	7·4	D	Sept. 2	12	38	37·2
755	B.D. −19° 4832	6·8	D	3	14	17	55·3
780	54 Sagittarii	5·4	D	5	11	44	27·9
781	55 Sagittarii	5·1	D	5	13	06	35·5
794	17 Aquarii	6·2	D	7	15	51	14·2
801	19 Arietis	6·0	R	13	13	35	20·4
848	B.D. −18° 4986	7·0	D	Oct. 1	9	46	56·0
849	B.D. −19° 5047	Var.	D	1	10	00	50·5
862	B.D. −17° 5611	6·9	D	2	11	30	48·1
996	ζ Tauri	3·0	D	Nov. 10	14	52	17·5
1005	74 Geminorum	5·2	R	12	15	23	43·7

TABLE II

Serial No.	Coeff. of $\Delta\alpha$	Coeff. of $\Delta\delta$	p	q	p^2	pq	q^2	$\Delta\sigma$	$p\Delta\sigma$	$q\Delta\sigma$	Lun.
135	+14	−0·2	+100	− 1	100	−100	0	−1·0	−1·0	0·0	187
195	+13	−0·4	+ 96	−29	92	− 28	8	−2·8	−2·7	+0·8	188
214	+ 9	−0·8	+ 83	−56	69	− 46	31	+0·2	+0·2	−0·1	188
304	+13	+0·5	+ 66	+75	44	+ 50	56	−1·2	−0·8	−0·9	189
346	+12	−0·6	+ 70	−72	49	− 50	51	+0·4	+0·3	−0·3	189
346	− 9	−0·7	− 79	−61	63	+ 48	37	+1·1	−0·9	−0·7	189
400	− 2	−1·0	+ 23	−97	5	− 22	95	+1·6	+0·4	−1·6	190
406	+14	−0·4	+100	− 3	100	− 3	0	−1·4	−1·4	0·0	190
481	+ 4	−1·0	+ 56	−83	31	− 46	69	+2·2	+1·2	−1·8	191
506	+14	−0·2	+ 99	−11	99	− 11	1	−1·3	−1·3	+0·1	191
506	−14	+0·3	− 98	+18	97	− 18	3	+0·8	−0·8	+0·1	191
507	+14	−0·2	+100	−10	99	− 10	1	−1·2	−1·2	+0·1	191
562	+14	0·0	+ 99	+17	97	+ 17	3	−1·4	−1·4	−0·2	192
564	+14	−0·1	+ 99	+13	98	+ 13	2	−1·3	−1·3	−0·2	192
750	+11	−0·6	+ 81	−59	65	− 48	35	+0·2	+0·2	−0·1	194
755	+14	−0·1	+ 98	−18	97	− 18	3	−0·4	−0·4	+0·1	194
780	+ 4	+1·0	+ 46	+89	21	+ 41	79	−2·4	−1·1	−2·1	194
781	+ 4	+1·0	+ 47	+88	22	+ 41	78	−1·8	−0·8	−1·6	194
794	+12	+0·6	+ 95	+32	90	+ 30	10	−0·8	−0·8	−0·3	194
801	−14	−0·1	− 99	+16	97	− 16	3	+0·2	−0·2	0·0	194
848	+10	+0·7	+ 81	+59	65	+ 48	35	−2·4	−1·9	−1·4	195
849	+14	−0·3	+ 92	−38	85	− 35	15	−2·1	−1·9	+0·8	195
862	+13	−0·4	+ 81	−59	65	− 48	35	−1·2	−1·0	+0·7	195
996	+13	+0·3	+ 95	+32	90	+ 30	10	−1·9	−1·8	−0·6	196
1005	− 7	+0·9	− 66	+75	44	− 50	56	+1·4	−0·9	+1·0	196